Lab 3. Random variables.

sgr. 3

1. Consider the random variables X and Y with the following distributions. Determine the distributions of $\underbrace{X+Y, 2X-3Y, X^3, 1/Y}_{\text{random variables}}$ and compute E[X+2Y], V[X-Y],

X and Y are independent.

$$X:\begin{pmatrix} -2 & 0 & 1 & 2 \\ \frac{1}{3} & \frac{1}{12} & \frac{1}{4} & \frac{1}{3} \end{pmatrix}, \quad Y:\begin{pmatrix} -1 & 1 & 2 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}.$$

$$\boxed{X+Y}:\begin{pmatrix} -2+(-1) & -2+1 & -2+2 & 0+(-1) & 0+1 & 0+2 & 1+(-1) & 1+1 & 1+2 & 2+(-1) & 2+1 & 2+2 \\ \frac{1}{3}\cdot\frac{1}{3} & \frac{1}{3}\cdot\frac{1}{2} & \frac{1}{3}\cdot\frac{1}{6} & \frac{1}{12}\cdot\frac{1}{3} & \frac{1}{12}\cdot\frac{1}{2} & \frac{1}{12}\cdot\frac{1}{6} & \frac{1}{4}\cdot\frac{1}{3} & \frac{1}{4}\cdot\frac{1}{2} & \frac{1}{4}\cdot\frac{1}{6} & \frac{1}{3}\cdot\frac{1}{3} & \frac{1}{3}\cdot\frac{1}{2} & \frac{1}{3}\cdot\frac{1}{6} \end{pmatrix}$$

$$X+Y:\begin{pmatrix} -3 & -1 & 0 & -1 & 1 & 2 & 0 & 2 & 3 & 1 & 3 & 4 \\ \frac{1}{9} & \frac{1}{6} & \frac{1}{18} & \frac{1}{36} & \frac{1}{24} & \frac{1}{72} & \frac{1}{12} & \frac{1}{8} & \frac{1}{24} & \frac{1}{9} & \frac{1}{6} & \frac{1}{18} \end{pmatrix}, \quad X+Y:\begin{pmatrix} -3 & -1 & 0 & 1 & 2 & 3 & 4 \\ \frac{1}{9} & \frac{7}{36} & \frac{5}{36} & \frac{11}{72} & \frac{5}{36} & \frac{5}{24} & \frac{1}{18} \end{pmatrix}$$

$P(X+Y=-1)=\frac{1}{6}+\frac{1}{36}=\frac{7}{36}$; $P(X+Y=0)=\frac{1}{18}+\frac{1}{12}=\frac{5}{36}$

$\frac{1}{24}+\frac{1}{9}=\frac{11}{72}$; $\frac{1}{72}+\frac{1}{8}=\frac{10}{72}=\frac{5}{36}$; $\frac{1}{24}+\frac{1}{6}=\frac{5}{24}$

$$\boxed{2X}:\begin{pmatrix} -4 & 0 & 2 & 4 \\ \frac{1}{3} & \frac{1}{12} & \frac{1}{4} & \frac{1}{3} \end{pmatrix}, \quad \boxed{-3Y}:\begin{pmatrix} 3 & -3 & -6 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}$$

$$-3Y:\begin{pmatrix} -6 & -3 & 3 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \end{pmatrix}$$

$$X:\begin{pmatrix} -2 & 0 & 1 & 2 \\ \frac{1}{3} & \frac{1}{12} & \frac{1}{4} & \frac{1}{3} \end{pmatrix}, \quad X^3:\begin{pmatrix} -8 & 0 & 1 & 8 \\ \frac{1}{3} & \frac{1}{12} & \frac{1}{4} & \frac{1}{3} \end{pmatrix} \qquad Y:\begin{pmatrix} -1 & 1 & 2 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}$$

$$1/Y:\begin{pmatrix} -1 & 1 & 1/2 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{pmatrix}, \quad 1/Y:\begin{pmatrix} -1 & 1/2 & 1 \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \end{pmatrix}$$

$$X/Y:\begin{pmatrix} -2/(-1) & \frac{-2}{1} & \frac{-2}{2} & \frac{0}{-1} & \frac{0}{1} & \frac{0}{2} & \frac{1}{-1} & \frac{1}{1} & \frac{1}{2} & \dots \\ \frac{1}{3}\cdot\frac{1}{3} & \frac{1}{3}\cdot\frac{1}{2} & \frac{1}{3}\cdot\frac{1}{6} & \frac{1}{12}\cdot\frac{1}{3} & \frac{1}{12}\cdot\frac{1}{2} & \frac{1}{12}\cdot\frac{1}{6} & \dots \end{pmatrix} \quad \dots$$

$$E[X+2Y] = E[X] + E[2Y] = E[X] + 2\cdot E[Y] = \frac{1}{4} + 2\cdot\frac{1}{2} = \frac{1}{4} + 1 = \frac{5}{4}$$

$X+2Y \to$ distribution

$$E[X] = \cancel{-2\cdot\frac{1}{3}} + 0\cdot\frac{1}{12} + 1\cdot\frac{1}{4} + \cancel{2\cdot\frac{1}{3}} = \frac{1}{4}$$

$$E[Y] = (-1)\cdot\frac{1}{3} + 1\cdot\frac{1}{2} + \cancel{2}\cdot\frac{1}{\cancel{6}_{3}} = \cancel{-\frac{1}{3}} + \frac{1}{2} + \cancel{\frac{1}{3}} = \frac{1}{2}$$

$$V[X-Y] \overset{X,Y \text{ are indep.}}{=} V[X]+V[-Y] = V[X]+(-1)^2\cdot V[Y] = V[X]+V[Y] = \frac{35}{16} +$$

$$X-Y = X+(-Y), \quad V[c\cdot Y] = c^2 V[Y]$$

$$V[X] = E[X^2]-E^2[X] = \frac{9}{4}-\left(\frac{1}{4}\right)^2 = \frac{9}{4}-\frac{1}{16} = \frac{35}{16}$$

$$X^2 : \begin{pmatrix} (-2)^2 & 0^2 & 1^2 & 2^2 \\ \frac{1}{3} & \frac{1}{12} & \frac{1}{4} & \frac{1}{3} \end{pmatrix}, \quad X^2 : \begin{pmatrix} 0 & 1 & 4 \\ \frac{1}{12} & \frac{1}{4} & \frac{2}{3} \end{pmatrix}, \quad \underline{E[X^2]} = 0\cdot\frac{1}{12}+1\cdot\frac{1}{4}+4\cdot\frac{2}{3} = \frac{1}{4}+\frac{8}{3} = \frac{27}{12} = \underline{\frac{9}{4}}$$

$$V[Y] = \underbrace{E[Y^2]}_{\frac{3}{2}} - E^2[Y] = \frac{3}{2}-\frac{1}{4} = \frac{5}{4}$$

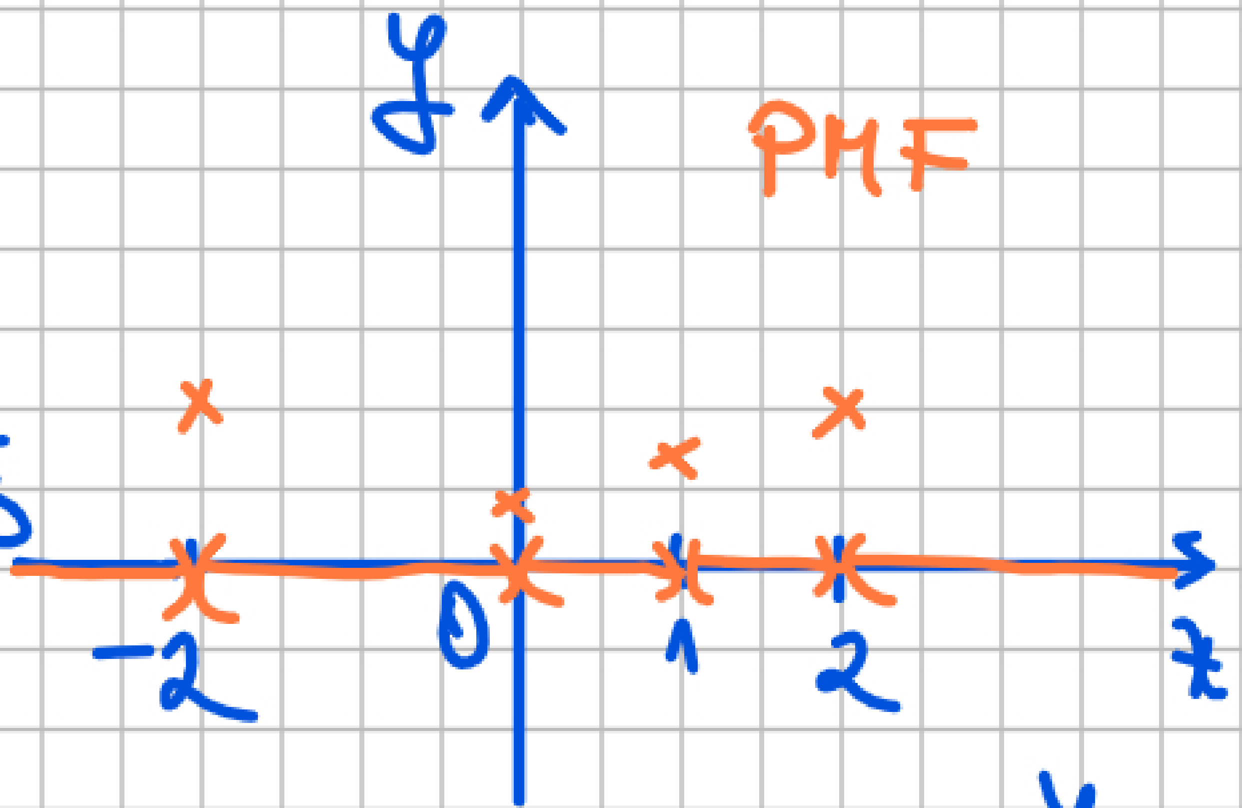
$$Std[Y] = \sqrt{V[Y]} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$$

2. Determine the PMF and CDF of the random variables X and Y from the previous exercise and plot them.

$$X: \begin{pmatrix} -2 & 0 & 1 & 2 \\ \frac{1}{3} & \frac{1}{12} & \frac{1}{4} & \frac{1}{3} \end{pmatrix}$$

PMF: $f:\mathbb{R} \to [0,1]$, $f(-2)=\frac{1}{3}$, $f(0)=\frac{1}{12}$, $f(1)=\frac{1}{4}$, $f(2)=\frac{1}{3}$

$f(x)=0$, $\forall x \notin \{-2,0,1,2\}$

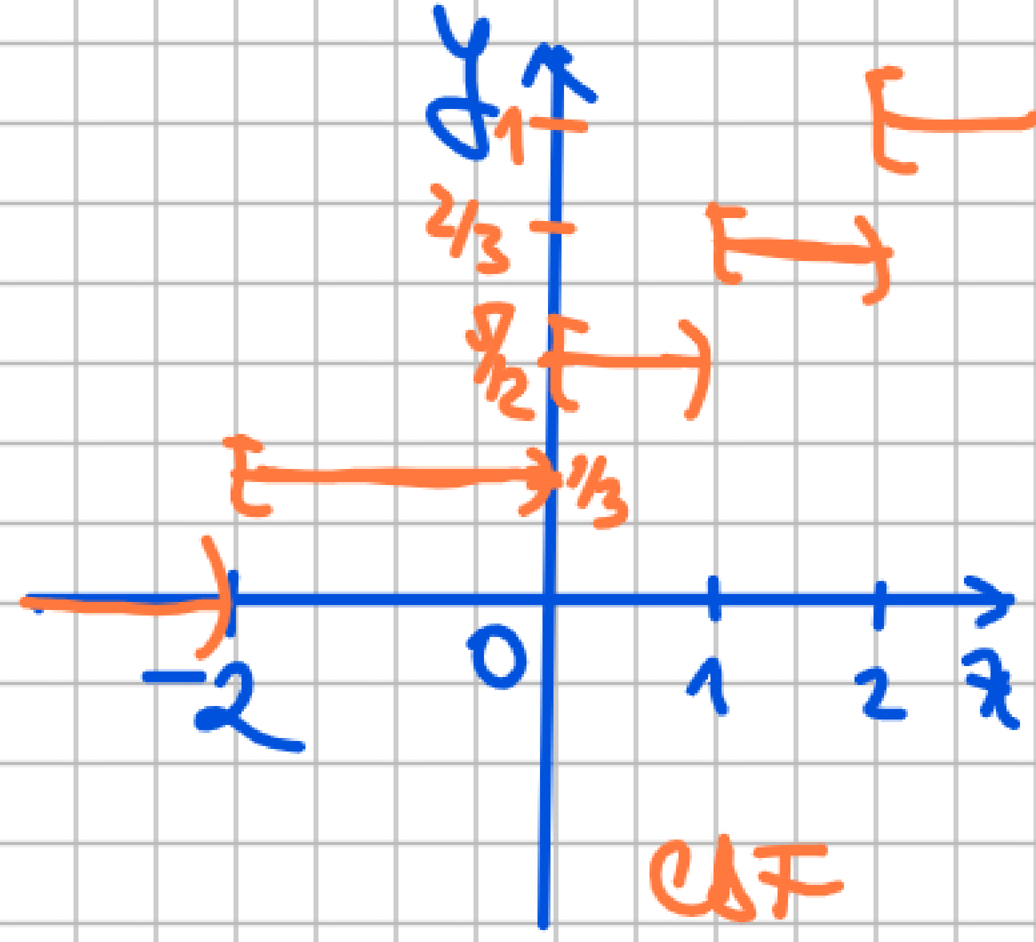


CDF: $F:\mathbb{R}\to[0,1]$,

$$F(x)=\begin{cases} 0, & x<-2 \\ \frac{1}{3}, & -2\le x<0 \\ \frac{1}{3}+\frac{1}{12}, & 0\le x<1 \\ \frac{1}{3}+\frac{1}{12}+\frac{1}{4}, & 1\le x<2 \\ 1, & x\ge 2 \end{cases} = \begin{cases} 0, & x<-2 \\ \frac{1}{3}, & -2\le x<0 \\ \frac{5}{12}, & 0\le x<1 \\ \frac{2}{3}, & 1\le x<2 \\ 1, & x\ge 2 \end{cases}$$

$F(x)=P(X\le x)$

$P(X\le 4)=P(X=-2)+P(X=0)+$
$+P(X=1)+P(X=2)$



PMF, CDF of Y $\to$ homework

Conditioning on evidence

1. ⓢ A spam filter is designed by looking at commonly occurring phrases in spam. Suppose that 80% of email is spam. In 10% of the spam emails, the phrase "free money" is used, whereas this phrase is only used in 1% of non-spam emails. A new email has just arrived, which does mention "free money". What is the probability that it is spam?

S: the email is spam, F: the email contains "free money"

$P(S)=0.8$

$P(F|S)=0.1$, $P(F|\bar{S})=0.01$

$P(S|F)=?$

$$P(S|F)=\frac{P(F|S)\cdot P(S)}{P(F)}=\frac{0.1\cdot 0.8}{0.082}=\frac{0.08}{0.082}=\frac{80}{82}=\frac{40}{41}\approx 0.978$$

$$P(F)=P(F|S)\cdot P(S)+P(F|\bar{S})\cdot P(\bar{S})=0.1\cdot 0.8+0.01\cdot 0.2=0.08+0.002=0.082$$

$$P(\bar{S})=1-P(S)=1-0.8=0.2$$

3. According to the CDC (Centers for Disease Control and Prevention), men who smoke are 23 times more likely to develop lung cancer than men who don't smoke. Also according to the CDC, 21.6% of men in the U.S. smoke. What is the probability that a man in the U.S. is a smoker, given that he develops lung cancer?

S: the man smokes $\quad P(S) = 0.216$

C: man has lung cancer

$P(C|S) = 23 \cdot P(C|\bar{S})$

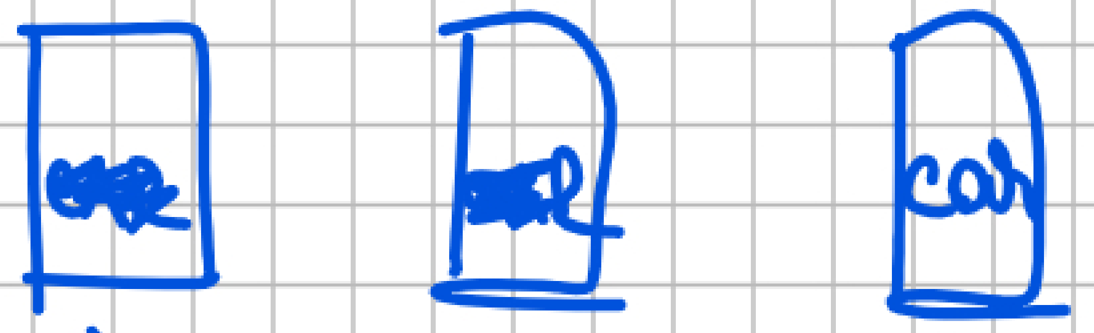
$P(S|C) = ?$

$$P(S|C) = \frac{P(C|S)P(S)}{P(C)} = \frac{P(C|S) \cdot P(S)}{P(C|S) \cdot P(S) + P(C|\bar{S}) \cdot P(\bar{S})} = \frac{23 \cdot P(C|\bar{S}) \cdot P(S)}{23 \cdot \underline{P(C|\bar{S})} \cdot P(S) + \underline{P(C|\bar{S})} \cdot P(\bar{S})} =$$

$$= \frac{23 \cdot \cancel{P(C|\bar{S})} \cdot P(S)}{\cancel{P(C|\bar{S})}\,(23P(S) + P(\bar{S}))} = \frac{23 \cdot P(S)}{23P(S) + 1 - P(S)} = \frac{23P(S)}{22P(S)+1} = \frac{23 \cdot 0.216}{22 \cdot 0.216 + 1} = 0.864$$

Assignment 1. Code the Monty Hall pb. in R.

Estimate the probability of winning if the contestant switches the door and the probab. of winning if he/she doesn't switch.



D1 D2 D3

car = sample(1:3,1)

ws

wns = wns+1

C1. D1 → car

wns = wns+1

C2. D2 → car

ws = ws+1

C3. D3 → car

ws = ws+1

ws/n →

wns/n →

6. A hat contains 100 coins, where 99 are fair but one is double-headed (always landing Heads). A coin is chosen uniformly at random. The chosen coin is flipped 7 times, and it lands Heads all 7 times. Given this information, what is the probability that the chosen coin is double-headed? (Of course, another approach here would be to *look at both sides of the coin*—but this is a metaphorical coin.)

sgr. 2

1. Consider the random variables X and Y with the following distributions. Determine the distributions of X+Y, 2Y, Y^2, X*Y , 1/Y

and compute E[X], E[Y], E[X-3Y], V[3X+2Y] if X and Y are independent.

$$X: \begin{pmatrix} \frac{1}{2} & 1 & 2 \\ 0.2 & 0.3 & 0.5 \end{pmatrix}, \quad Y: \begin{pmatrix} -1 & 1 & 2 \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{pmatrix}$$

$$X+Y: \begin{pmatrix} \frac{1}{2}+(-1) & \frac{1}{2}+1 & \frac{1}{2}+2 & 1+(-1) & 1+1 & 1+2 & 2+(-1) & 2+1 & 2+2 \\ 0.2\cdot\frac{1}{2} & 0.2\cdot\frac{1}{6} & 0.2\cdot\frac{1}{3} & 0.3\cdot\frac{1}{2} & 0.3\cdot\frac{1}{6} & 0.3\cdot\frac{1}{3} & 0.5\cdot\frac{1}{2} & 0.5\cdot\frac{1}{6} & 0.5\cdot\frac{1}{3} \end{pmatrix}$$

$$X+Y: \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} & \frac{5}{2} & 0 & 2 & \boxed{3} & 1 & \boxed{3} & 4 \\ \frac{1}{10} & \frac{1}{30} & \frac{2}{30} & \frac{3}{20} & \frac{1}{20} & \frac{1}{10} & \frac{1}{4} & \frac{1}{12} & \frac{1}{6} \end{pmatrix}, \quad \boxed{X+Y}: \begin{pmatrix} -\frac{1}{2} & 0 & 1 & \frac{3}{2} & 2 & \frac{5}{2} & 3 & 4 \\ \frac{1}{10} & \frac{3}{20} & \frac{1}{4} & \frac{1}{30} & \frac{1}{20} & \frac{2}{3} & \frac{11}{60} & \frac{1}{6} \end{pmatrix}$$

$${}^{6)}\frac{1}{10} + {}^{5)}\frac{1}{12} = \frac{11}{60}$$

$${}^{3)}\frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$$

$$\boxed{2\cdot Y}: \begin{pmatrix} -2 & 2 & 4 \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{pmatrix}, \quad Y^2: \begin{pmatrix} (-1)^2 & 1^2 & 2^2 \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{pmatrix}, \quad \boxed{Y^2}: \begin{pmatrix} 1 & 4 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$X : \begin{pmatrix} \frac{1}{2} & 1 & 2 \\ 0.2 & 0.3 & 0.5 \end{pmatrix} \qquad Y : \begin{pmatrix} -1 & 1 & 2 \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{pmatrix}$$

$$X \cdot Y : \begin{pmatrix} \frac{1}{2}\cdot(-1) & \frac{1}{2}\cdot 1 & \frac{1}{2}\cdot 2 & 1\cdot(-1) & 1\cdot 1 & 1\cdot 2 & \ldots \\ 0.2\cdot\frac{1}{2} & 0.2\cdot\frac{1}{6} & 0.2\cdot\frac{1}{3} & 0.3\cdot\frac{1}{2} & & \ldots & \end{pmatrix}$$

$$1/Y : \begin{pmatrix} \frac{1}{-1} & \frac{1}{1} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{pmatrix} , \quad 1/Y : \begin{pmatrix} -1 & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \end{pmatrix}$$

$$E[X] = \frac{1}{2}\cdot 0.2 + 1\cdot 0.3 + 2\cdot 0.5 = 0.1 + 0.3 + 1 = \underline{\underline{1.4}}$$

$$E[Y] = -1\cdot\frac{1}{2} + 1\cdot\frac{1}{6} + 2\cdot\frac{1}{3} = \overset{3/}{-\frac{1}{2}} + \frac{1}{6} + \overset{2/}{\frac{2}{3}} = \frac{2}{6} = \frac{1}{3}$$

$$E[X-3Y] = E[X] + E[-3Y] = E[X] + (-3)\cdot E[Y] = 1.4 - 3\cdot\frac{1}{3} = 1.4 - 1 = 0.4$$

$$V[3X+2Y] = V[3X] + V[2Y] = 3^2\cdot V[X] + 2^2\cdot V[Y] = 9\cdot V[X] + 4\cdot V[Y]$$

$$X: \begin{pmatrix} \frac{1}{2} & 1 & 2 \\ 0.2 & 0.3 & 0.5 \end{pmatrix}, \quad X^2: \begin{pmatrix} \frac{1}{4} & 1 & 4 \\ 0.2 & 0.3 & 0.5 \end{pmatrix} \qquad Y: \begin{pmatrix} -1 & 1 & 2 \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{pmatrix}$$

$$V[X] = E[X^2] - E^2[X] = 2.35 - 1.4^2 = 0.39 > 0$$

$$E[X^2] = \frac{47}{20} = 2.35$$

$$Std[X] = \sqrt{V[X]} = \sqrt{0.39} = 0.624$$

$$V[Y] = E[Y^2] - E^2[Y] = \boxed{2} - \left(\frac{1}{3}\right)^2 = \overset{9}{2} - \frac{1}{9} = \frac{17}{9} = 1.89$$

$$V[3X + 2Y] = 9V[X] + 4V[Y] = 9 \cdot 0.39 + 4 \cdot 1.89 = 7.07$$

2. Determine the PMF and CDF of X and Y from the previous exercise and plot them.

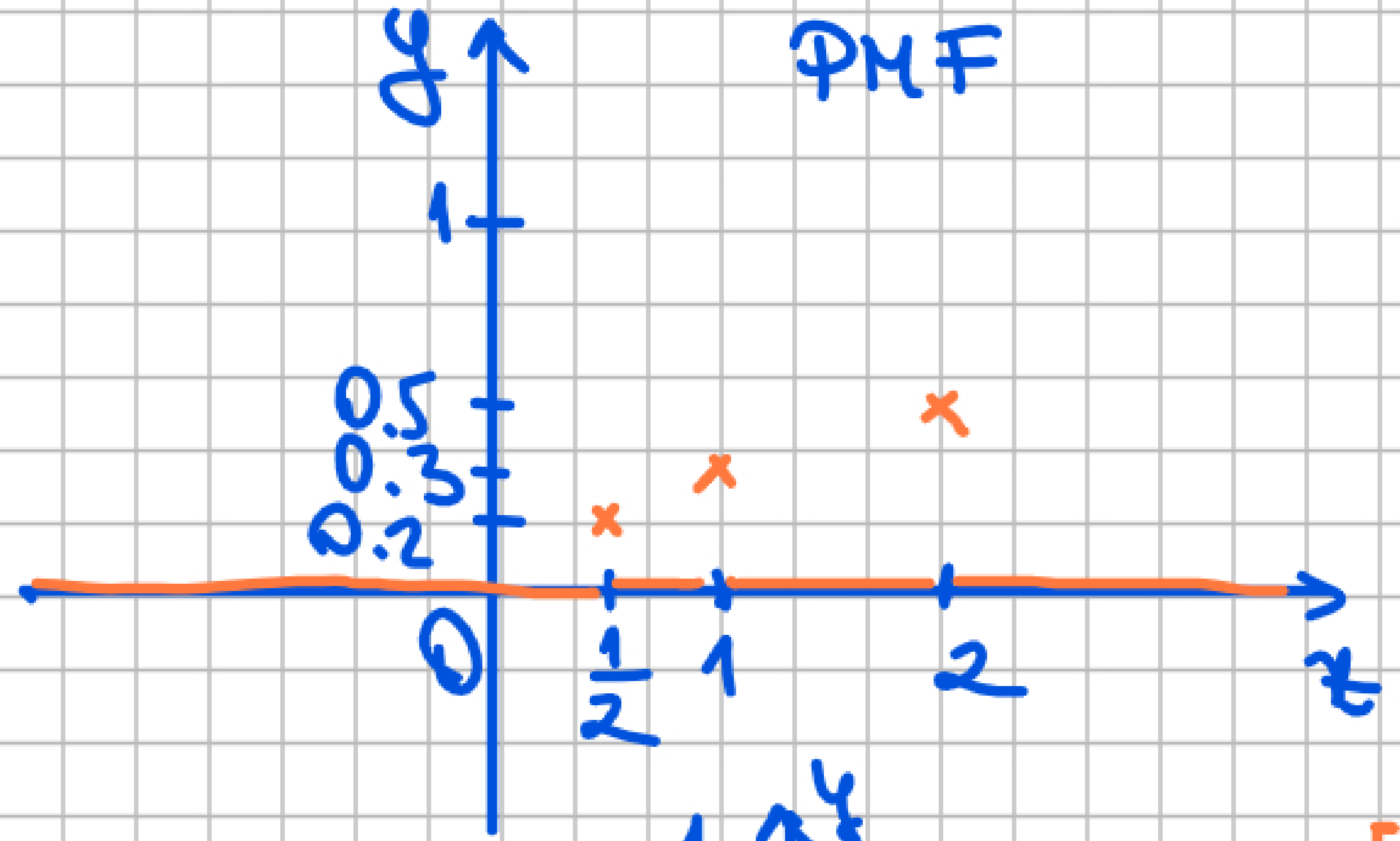
$$X: \begin{pmatrix} \frac{1}{2} & 1 & 2 \\ 0.2 & 0.3 & 0.5 \end{pmatrix}$$

PMF : $f:\mathbb{R}\to[0,1]$, $f(\frac{1}{2})=P(X=\frac{1}{2})=0.2$

$f(1)=0.3$

$f(2)=0.5$

$f(x)=0, (\forall)\ x\notin\{\frac{1}{2},1,2\}$



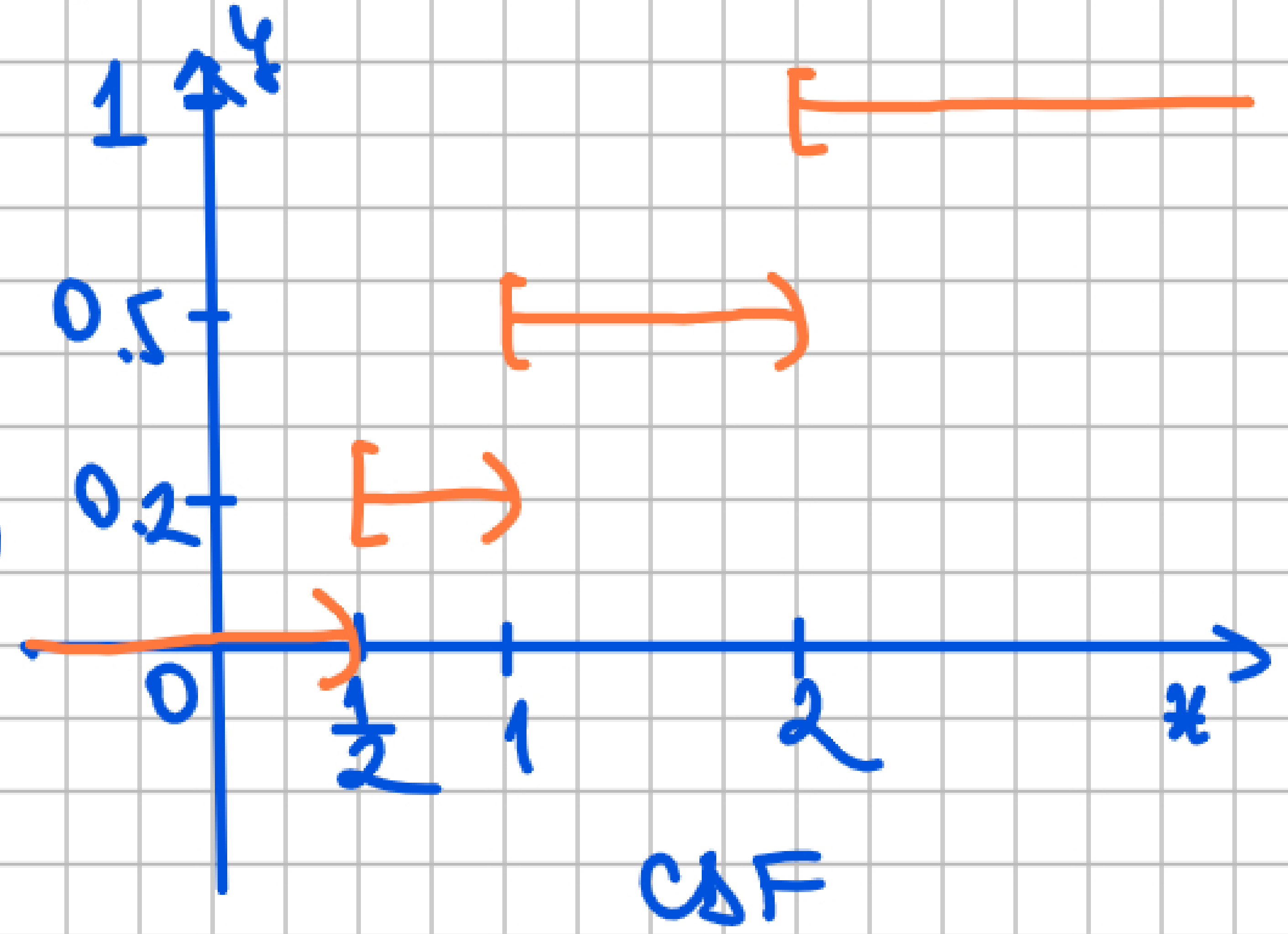
CDF : $F:\mathbb{R}\to[0,1]$,

$F(0)=P(X\le 0)=0$

$F(\frac{3}{4})=P(X\le\frac{3}{4})=P(X=\frac{1}{2})=0.2$

$F(\underset{\frac{3}{2}}{x})=P(X\le x)=P(X=\frac{1}{2})+P(X=1)=0.2+0.3=0.5$

$$F(x)=\begin{cases} 0, & x<\frac{1}{2} \\ 0.2, & \frac{1}{2}\le x<1 \\ 0.2+0.3, & 1\le x<2 \\ 0.2+0.3+0.5, & x\ge 2 \end{cases} = \begin{cases} 0, & x<\frac{1}{2} \\ 0.2, & x\in[\frac{1}{2},1) \\ 0.5, & x\in[1,2) \\ 1, & x\ge 2 \end{cases}$$



2. Ⓢ A woman is pregnant with twin boys. Twins may be either identical or fraternal. Suppose that 1/3 of twins born are identical, that identical twins have a 50% chance of being both boys and a 50% chance of being both girls, and that for fraternal twins each twin independently has a 50% chance of being a boy and a 50% chance of being a girl. Given the above information, what is the probability that the woman's twins are identical?

B : the woman is pregnant with twin boys

I : the twins are identical

$$P(I/B) = \frac{P(B|I)\cdot P(I)}{P(B)} = \frac{P(B|I)P(I)}{P(B|I)\cdot P(I) + P(B|\bar{I})\cdot P(\bar{I})} = \frac{\frac{1}{2}\cdot\frac{1}{3}}{\frac{1}{2}\cdot\frac{1}{3} + \frac{1}{\cancel{4}_{2}}\cdot\frac{\cancel{2}}{3}} = \frac{\frac{1}{6}}{\frac{1}{6}+\frac{1}{6}} = \frac{\frac{1}{6}}{\frac{2}{6}} = \frac{1}{2}$$

$$P(I) = \frac{1}{3} \Rightarrow P(\bar{I}) = 1 - P(I) = \frac{2}{3}$$

$$P(B|I) = \frac{1}{2}$$

$$P(B|\bar{I}) = \frac{1}{4}$$

11. An *exit poll* in an election is a survey taken of voters just after they have voted. One major use of exit polls has been so that news organizations can try to figure out as soon as possible who won the election, before the votes are officially counted. This has been notoriously inaccurate in various elections, sometimes because of *selection bias*: the sample of people who are invited to and agree to participate in the survey may not be similar enough to the overall population of voters.

Consider an election with two candidates, Candidate A and Candidate B. Every voter is invited to participate in an exit poll, where they are asked whom they voted for; some accept and some refuse. For a randomly selected voter, let A be the event that they voted for A, and W be the event that they are willing to participate in the exit poll. Suppose that $P(W|A) = 0.7$ but $P(W|A^c) = 0.3$. In the exit poll, 60% of the respondents say they voted for A (assume that they are all honest), suggesting a comfortable victory for A. Find $P(A)$, the true proportion of people who voted for A.

→ homework

Exercise 24. Bag A contains 3 red balls and 7 blue balls. Bag B contains 8 red balls and 4 blue balls. Bag C contains 5 red balls and 11 blue balls. A bag is chosen at random, with each bag being equally likely to be chosen, and then a ball is chosen at random from that bag. Calculate the probabilities that: (a) a red ball is chosen; (b) a blue ball is chosen; (c) a red ball from bag B is chosen. If it is known that a red ball is chosen, what is the probability that it comes from bag A? If it is known that a blue ball is chosen, what is the probability that it comes from bag B?

A : bag A is chosen
B : bag B is chosen
C : bag C is chosen

R: a red ball is chosen

a) $P(R) = P(R|A)\cdot P(A) + P(R|B)P(B) + P(R|C)P(C)$ (law of total probability)

$R|A$: a red ball is chosen from bag A

$P(R|A) = \frac{3}{10}$, $P(R|B) = \frac{8}{12} = \frac{2}{3}$, $P(R|C) = \frac{5}{16}$

$P(A) = P(B) = P(C) = \frac{1}{3}$

$P(R) = \frac{3}{10}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{1}{3} + \frac{5}{16}\cdot\frac{1}{3} = \frac{1}{3}\left({}^{24/}\frac{3}{10} + {}^{80/}\frac{2}{3} + {}^{15/}\frac{5}{16}\right) = \frac{1}{3}\cdot\frac{72+160+75}{240} = \frac{1}{3}\cdot\frac{307}{240} = \frac{307}{720} = 0.426$

b) BR: a blue ball is chosen

$P(BR) = 1 - 0.426 = 0.574$

$P(BR) = P(BR|A)\cdot P(A) + P(BR|B)P(B) + P(BR|C)\cdot P(C)$

$$P(A|R) = \frac{P(R|A)\cdot P(A)}{P(R)} = \frac{P(R|A)\cdot P(A)}{P(R|A)\cdot P(A)+P(R|B)\cdot P(B)+P(R|C)P(C)}$$

Bayes' rule

$$P(A|R) = \frac{\frac{\cancel{3}}{10}\cdot\frac{1}{\cancel{3}}}{\frac{307}{720}} = \frac{1}{1\cancel{0}}\cdot\frac{72\cancel{0}}{307} = \frac{72}{307} = 0.235$$

$$P(B|Br) = \frac{P(Br|B)\cdot P(B)}{P(Br)} = \frac{\frac{\cancel{4}}{3\cancel{12}}\cdot\frac{1}{3}}{1-\frac{307}{720}} = \frac{\frac{1}{9}}{\frac{413}{720}} = \frac{1}{\cancel{9}}\cdot\frac{\overset{80}{\cancel{720}}}{413} = \frac{80}{413} = 0.194$$

Monty Hall problem → implement in R

- $w\_s = 0$; $w\_ms = 0$

for

- $car = sample(1:3, 1)$
  - $car = 1 \rightarrow w\_ms = w\_ms + 1$
  - $car = 2 \rightarrow w\_s = w\_s + 1$
  - $car = 3 \rightarrow w\_s = w\_s + 1$

- w_s/n
  w_ms/n

sgr. 5

Exercise 29. The distribution law of a discrete random variable X is:

x_i	-4	-2	1	2	3
$P(X = x_i)$	0.10	0.35	0.15	0.25	0.15

$$Y : \begin{pmatrix} -2 & -1 & 0 \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{pmatrix}$$

Consider the random variable Y with the following distribution. Determine the distribution of X+Y, 3Y, X*Y, X^2, Y/X and compute V[X], E[Y], E[5X-2Y], P(-1<= X<3), P(Y>1).

$$X+Y : \begin{pmatrix} -4+(-2) & -4+(-1) & -4+0 & -2+(-2) & -2+(-1) & -2+0 & 1+(-2) & 1+(-1) & 1+0 & 2-2 & 2-1 & 2+0 \\ 0.10\cdot\frac{1}{5} & 0.10\cdot\frac{2}{5} & 0.10\cdot\frac{2}{5} & 0.35\cdot\frac{1}{5} & 0.35\cdot\frac{2}{5} & 0.35\cdot\frac{2}{5} & 0.15\cdot\frac{1}{5} & 0.15\cdot\frac{2}{5} & 0.15\cdot\frac{2}{5} & 0.25\cdot\frac{1}{5} & 0.25\cdot\frac{2}{5} & 0.25\cdot\frac{2}{5} \end{pmatrix}$$

$$\begin{pmatrix} 3+(-2) & 3+(-1) & 3+0 \\ 0.15\cdot\frac{1}{5} & 0.15\cdot\frac{2}{5} & 0.15\cdot\frac{2}{5} \end{pmatrix}$$

$$X+Y : \begin{pmatrix} -6 & -5 & -4 & -4 & -3 & -2 & -1 & 0 & 1 & 0 & 1 & 2 & 1 & 2 & 3 \\ 0.02 & 0.04 & \underbrace{0.04 \quad 0.07} & & 0.14 & 0.14 & 0.03 & 0.06 & 0.06 & 0.05 & 0.1 & 0.1 & 0.03 & 0.06 & 0.06 \end{pmatrix}$$

$$X+Y : \begin{pmatrix} -6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\ 0.02 & 0.04 & 0.11 & 0.14 & 0.14 & 0.03 & 0.11 & 0.19 & 0.16 & 0.06 \end{pmatrix}$$

$0.06+0.1+0.03=0.19$

$$Y: \begin{pmatrix} -2 & -1 & 0 \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{pmatrix}, \quad 3\cdot Y: \begin{pmatrix} 3\cdot(-2) & 3\cdot(-1) & 3\cdot 0 \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{pmatrix}$$

$$3\cdot Y: \begin{pmatrix} -6 & -3 & 0 \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{pmatrix}, \quad X^2: \begin{pmatrix} (-4)^2 & (-2)^2 & 1^2 & 2^2 & 3^2 \\ 0.1 & 0.35 & 0.15 & 0.25 & 0.15 \end{pmatrix}$$

$$X^2: \begin{pmatrix} 1 & 4 & 9 & 16 \\ 0.15 & 0.6 & 0.15 & 0.1 \end{pmatrix}$$

$$X\cdot Y: \begin{pmatrix} (-4)\cdot(-2) & (-4)\cdot(-1) & (-4)\cdot 0 & (-2)\cdot(-2) & (-2)\cdot(-1) & \dots \\ 0.10\cdot\frac{1}{5} & 0.10\cdot\frac{2}{5} & 0.10\cdot\frac{2}{5} & 0.35\cdot\frac{1}{5} & \dots & \end{pmatrix}$$

$$P(X\cdot Y=(-4)\cdot(-2)) = P(X=-4, Y=-2) = P(X=-4)\cdot P(Y=-2) = 0.10\cdot\frac{1}{5} = 0.02$$

$$P(X=1, X=1) = P(X=1) = 0.15, \quad P(X=-4, X=-2) = 0$$

$$A\cap A = A$$

$$Y/X: \begin{pmatrix} \frac{-2}{-4} & \frac{-2}{-2} & \frac{-2}{1} & \frac{-2}{2} & \frac{-2}{3} & \frac{-1}{-4} & \frac{-1}{-2} & \dots \\ \frac{1}{5}\cdot 0.1 & \frac{1}{5}\cdot 0.35 & \frac{1}{5}\cdot 0.15 & \frac{1}{5}\cdot 0.25 & \frac{1}{5}\cdot 0.15 & \frac{2}{5}\cdot 0.1 & \dots & \end{pmatrix}$$

$$E[X] = -4\cdot 0.1 + (-2)\cdot 0.35 + 1\cdot 0.15 + 2\cdot 0.25 + 3\cdot 0.15 =$$

$$= -0.4 - 0.7 + 0.15 + 0.5 + 0.45 = -1.1 + 1.1 = \underline{\underline{0}}$$

$$, E[X^2] = 5.5$$

$$E^2[X] = 5.5 - 0 = 5.5$$

$$Std[X] = \sqrt{V[X]} = \sqrt{5.5} = 2.35$$

$$E[5X - 2Y] = E[5X] + E[-2Y] = 5\cdot E[X] + (-2)E[Y] = 5\cdot 0 + (-2)\cdot\left(-\frac{4}{5}\right) = \frac{8}{5}$$

$$E[Y] = -\frac{4}{5}$$

$$V[X+2Y] = V[X] + V[2Y] = V[X] + 2^2 \cdot V[Y] = 5.5 + 4 \cdot 0.56 = 5.5 + 2.24 = 7.74$$

$$V[Y] = E[Y^2] - E^2[Y] = \frac{6}{5} - \left(\frac{4}{5}\right)^2 = \frac{6}{5} - \frac{16}{25} = \frac{14}{25} = 0.56$$

$$P(-1 \leq X < 3) = P(X=1) + P(X=2) = 0.15 + 0.25 = 0.4$$

$$P(Y>1) = 0$$

2. Determine the PMF and CDF of X and Y given in the previous example and plot them.

PMF : $f:\mathbb{R}\to[0,1]$

$f(x)=P(X=x)$

$f(-4)=P(X=-4)=0.1$

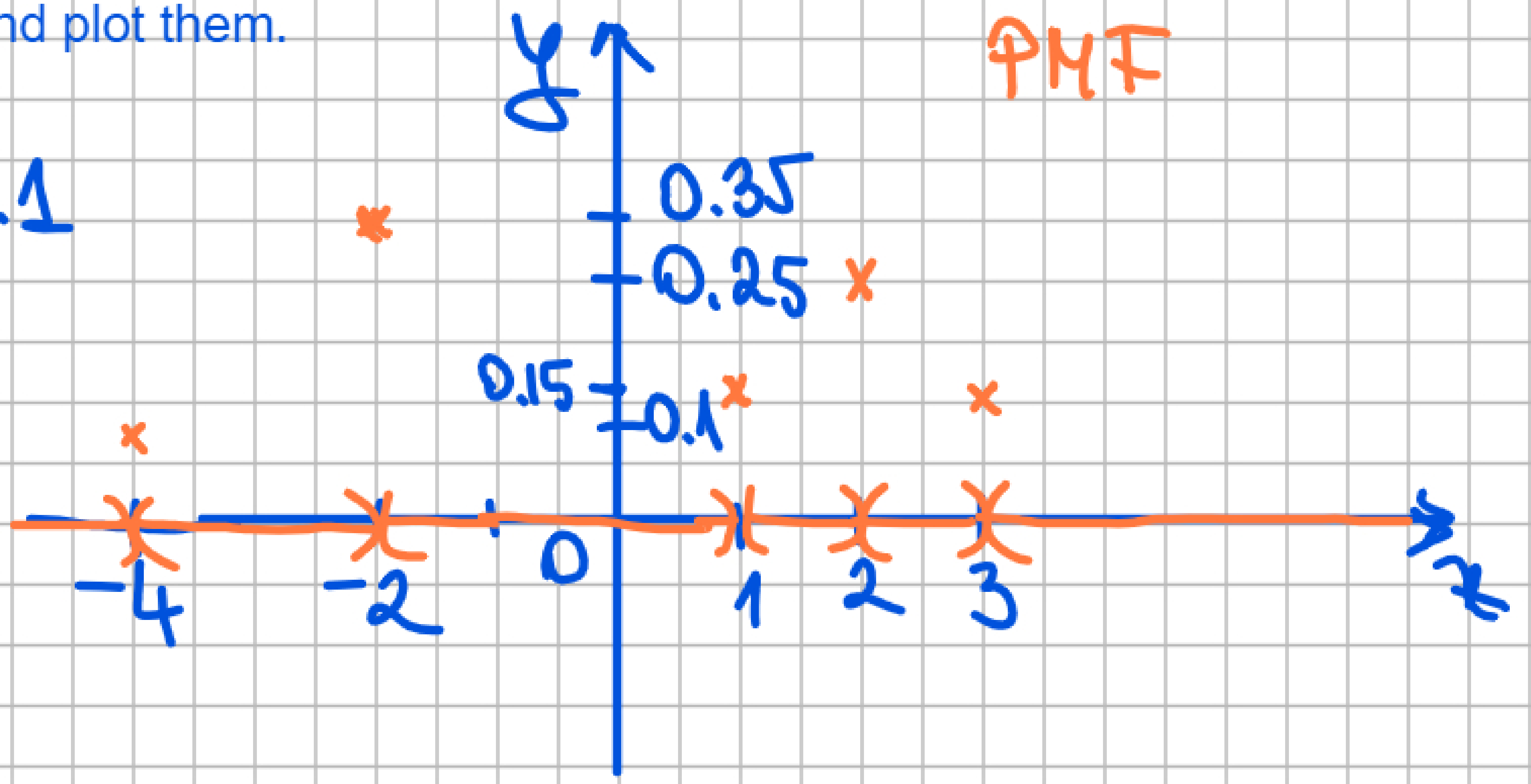
$f(-2)=0.35$

$f(1)=0.15$

$f(2)=0.25$

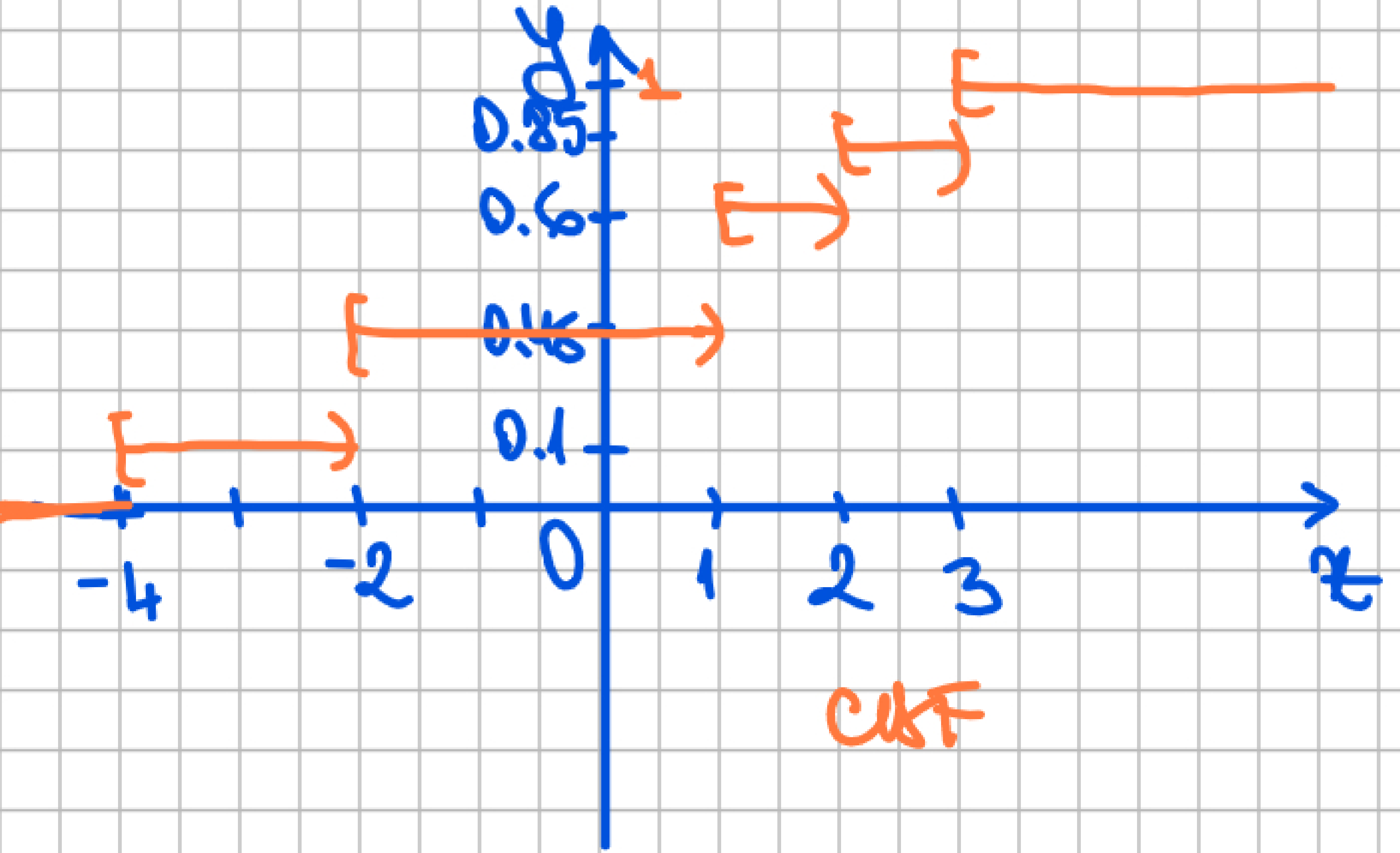
$f(3)=0.15$

$f(x)=0,\ \forall\, x\notin\{-4,-2,1,2,3\}$



CDF : $F:\mathbb{R}\to[0,1]$, $F(x)=P(X\le x)$

$$F(x)=\begin{cases} 0, & x<-4\\ 0.1, & -4\le x<-2\\ 0.1+0.35, & -2\le x<1\\ 0.1+0.35+0.15, & 1\le x<2\\ 0.1+0.35+0.15+0.25, & 2\le x<3\\ 1, & x\ge 3\end{cases} = \begin{cases} 0, & x<-4\\ 0.1, & -4\le x<-2\\ 0.45, & -2\le x<1\\ 0.6, & 1\le x<2\\ 0.85, & 2\le x<3\\ 1, & x\ge 3\end{cases}$$



PMF + CDF of Y → homework

Exercise 25. A doctor assumes that a patient has one of three diseases d1, d2, or d3. Before any test, he assumes an equal probability for each disease. He carries out a test that will be positive with probability 0.8 if the patient has d1, 0.6 if he has disease d2, and 0.4 if he has disease d3. Given that the outcome of the test was positive, what probabilities should the doctor now assign to the three possible diseases?

D_1 : the pacient has disease d1   T: the test is positive

D_2 : the pacient has disease d2

D_3 : —"— —"— d3

$P(D_1) = P(D_2) = P(D_3) = \frac{1}{3}$

$P(T/D_1) = 0.8$, $P(T|D_2) = 0.6$, $P(T/D_3) = 0.4$

$$P(D_1/T) \underset{\text{Bayes' rule}}{=} \frac{P(T|D_1)\cdot P(D_1)}{P(T)} \underset{\text{LOTP}}{=} \frac{P(T|D_1)\cdot P(D_1)}{P(T|D_1)\cdot P(D_1) + P(T/D_2)\cdot P(D_2) + P(T|D_3)\cdot P(D_3)} =$$

$$= \frac{0.8\cdot\frac{1}{3}}{0.8\cdot\frac{1}{3} + 0.6\cdot\frac{1}{3} + 0.4\cdot\frac{1}{3}} = \frac{0.8\cdot\cancel{\frac{1}{3}}}{\cancel{\frac{1}{3}}(0.8+0.6+0.4)} = \frac{0.8}{1.8} = \frac{8}{18} = \frac{4}{9} = 0.44$$

$$P(D_2 | T) = \frac{P(T|D_2) \cdot P(D_2)}{P(T)} = \frac{0.6 \cdot \frac{1}{3}}{\frac{1}{3} \cdot 1.8} = \frac{0.6}{1.8} = \frac{6}{18} = \frac{1}{3}$$

$$P(D_3 | T) = \frac{P(T|D_3) \cdot P(D_3)}{P(T)} = \frac{0.4 \cdot \frac{1}{3}}{\frac{1}{3} \cdot 1.8} = \frac{0.4}{1.8} = \frac{4}{18} = \frac{2}{9} = 0.22$$

BitPaper Share Knowledge

Exercise 27. The weather on a particular day is classified as either cold, warm or hot. There is a probability of 0.15 that it is cold and a probability of 0.25 that it is warm. In addition, on each day it may either rain or not rain. On cold days, there is a probability of 0.30 that it will rain, on warm days, there is a probability of 0.40 that it will rain and on hot days, there is a probability of 0.50 that it will rain. If it is not raining on a particular day, what is the probability that it is cold?

$$P(\bar{A}|B) = 1 - P(A|B)$$

C : the weather is cold

W : the weather is warm

H : the weather is hot

R : it rains on that day

$P(C) = 0.15$

$P(W) = 0.25$

$P(H) = 1 - (0.15 + 0.25) = 0.6$

$P(R|C) = 0.3$, $P(R/W) = 0.4$, $P(R/H) = 0.5$

$$P(C|\bar{R}) = \frac{P(\bar{R}|C)P(C)}{P(\bar{R})} = \frac{(1-P(R|C))P(C)}{1-P(R)} = \frac{0.7 \cdot 0.15}{0.15} = 0.7$$

$$P(\bar{R}) = 1 - P(R) = 1 - 0.85 = 0.15$$

$$P(R) = P(R|C) \cdot P(C) + P(R/W) \cdot P(W) + P(R|H) \cdot P(H) = 0.3 \cdot 0.15 + 0.4 \cdot 0.25 + 0.5 \cdot 0.6 =$$

(law of total probability)

$$= 0.45 + 0.1 + 0.3 = 0.85$$

6. Ⓢ *Benford's law* states that in a very large variety of real-life data sets, the first digit approximately follows a particular distribution with about a 30% chance of a 1, an 18% chance of a 2, and in general

$$P(D = j) = \log_{10}\left(\frac{j+1}{j}\right), \text{ for } j \in \{1, 2, 3, \ldots, 9\},$$

where D is the first digit of a randomly chosen element. Check that this is a valid PMF (using properties of logs, not with a calculator).

→ homework

Assignment : Implement/simulate the Monty Hall pb. in R.

Estimate the probability of winning if you use the switching strategy & don't use this strategy.

sgr. 6

1. Consider the random variables X and Y with the following distributions. Determine the distributions of X+Y, -2Y, X^2, X*Y, X/Y and compute E[X], V[X], E[2X-3Y], V[X+5Y], P(-1<=X<5), P(Y>1), X and Y are independent.

$$X: \begin{pmatrix} -2 & 0 & 1 \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{pmatrix}, \quad Y: \begin{pmatrix} -1 & 2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$X+Y: \begin{pmatrix} -2+(-1) & -2+2 & 0+(-1) & 0+2 & 1+(-1) & 1+2 \\ \frac{1}{2}\cdot\frac{1}{2} & \frac{1}{2}\cdot\frac{1}{2} & \frac{1}{6}\cdot\frac{1}{2} & \frac{1}{6}\cdot\frac{1}{2} & \frac{1}{3}\cdot\frac{1}{2} & \frac{1}{3}\cdot\frac{1}{2} \end{pmatrix}$$

$$X+Y: \begin{pmatrix} -3 & 0 & -1 & 2 & 0 & 3 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{12} & \frac{1}{12} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}, \quad \boxed{X+Y:} \begin{pmatrix} -3 & -1 & 0 & 2 & 3 \\ \frac{1}{4} & \frac{1}{12} & \frac{5}{12} & \frac{1}{12} & \frac{1}{6} \end{pmatrix}$$

$${}^{3)}\frac{1}{4} + {}^{2)}\frac{1}{6} = \frac{5}{12}$$

$$P(X=-2, Y=-1) = P(X=-2 \cap Y=-1) \overset{X,Y\text{-ind.}}{=\!=} P(X=-2)\cdot P(Y=-1) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

$$-2Y: \begin{pmatrix} (-2)(-1) & (-2)\cdot 2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad \boxed{-2Y:} \begin{pmatrix} -4 & 2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$X^2: \begin{pmatrix} (-2)^2 & 0^2 & 1^2 \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{pmatrix}, \quad \boxed{X^2:} \begin{pmatrix} 0 & 1 & 4 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}$$

$$X * Y: \begin{pmatrix} (-2)\cdot(-1) & (-2)\cdot 2 & 0\cdot(-1) & 0\cdot 2 & 1\cdot(-1) & 1\cdot 2 \\ \frac{1}{2}\cdot\frac{1}{2} & \frac{1}{2}\cdot\frac{1}{2} & \frac{1}{6}\cdot\frac{1}{2} & \frac{1}{6}\cdot\frac{1}{2} & \frac{1}{3}\cdot\frac{1}{2} & \frac{1}{3}\cdot\frac{1}{2} \end{pmatrix}$$

$$\overset{3/}{\frac{1}{4}} + \overset{2/}{\frac{1}{6}} = \frac{5}{12}$$

$$\boxed{X * Y:} \begin{pmatrix} -4 & -1 & 0 & 2 \\ \frac{1}{4} & \frac{1}{6} & \frac{1}{6} & \frac{5}{12} \end{pmatrix}$$

$$X/Y: \begin{pmatrix} \frac{-2}{-1} & \frac{-2}{2} & \frac{0}{-1} & \frac{0}{2} & \frac{1}{-1} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{12} & \frac{1}{12} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}, \quad \boxed{X/Y:} \begin{pmatrix} -1 & 0 & \frac{1}{2} & 2 \\ \frac{5}{12} & \frac{1}{6} & \frac{1}{6} & \frac{1}{4} \end{pmatrix}$$

$$X: \begin{pmatrix} -2 & 0 & 1 \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{pmatrix} \qquad E[X] = -\cancel{2}\cdot\frac{1}{\cancel{2}} + 0\cdot\frac{1}{6} + 1\cdot\frac{1}{3} = -1 + \frac{1}{3} = -\frac{2}{3}$$

$$V[X] = E[X^2] - E^2[X] = \frac{7}{3} - \left(-\frac{2}{3}\right)^2 = \frac{7}{3} - \frac{4}{9} = \frac{17}{9}$$

$$E[2X-3Y] = E[2X] + E[-3Y] = 2\cdot E[X] + (-3)\cdot E[Y] = 2\cdot\left(-\frac{2}{3}\right) + (-3)\cdot\frac{1}{2} = -\frac{4}{3} - \frac{3}{2} = \frac{-17}{6}$$

$$V[X+5Y] \underset{\substack{\uparrow \\ X,Y \text{ ind.}}}{=} V[X] + V[5Y] = V[X] + 5^2 V[Y] = \frac{17}{9} + 25\cdot\frac{9}{4} = \frac{17}{9} + \frac{225}{4} = \frac{2093}{36}$$

$$V[Y] = E[Y^2] - E^2[Y] = \frac{5}{2} - \frac{1}{4} = \frac{9}{4}$$

$$P(-1 \leq X < 5) = P(X=0) + P(X=1) = \frac{1}{6} + \frac{1}{3} = \frac{3}{6} = \frac{1}{2}$$

$$P(Y>1) = P(Y=2) = \frac{1}{2}$$

2. Determine the PMF and CDF of the random variables X and Y and plot them.

PMF: $f:\mathbb{R}\to[0,1]$

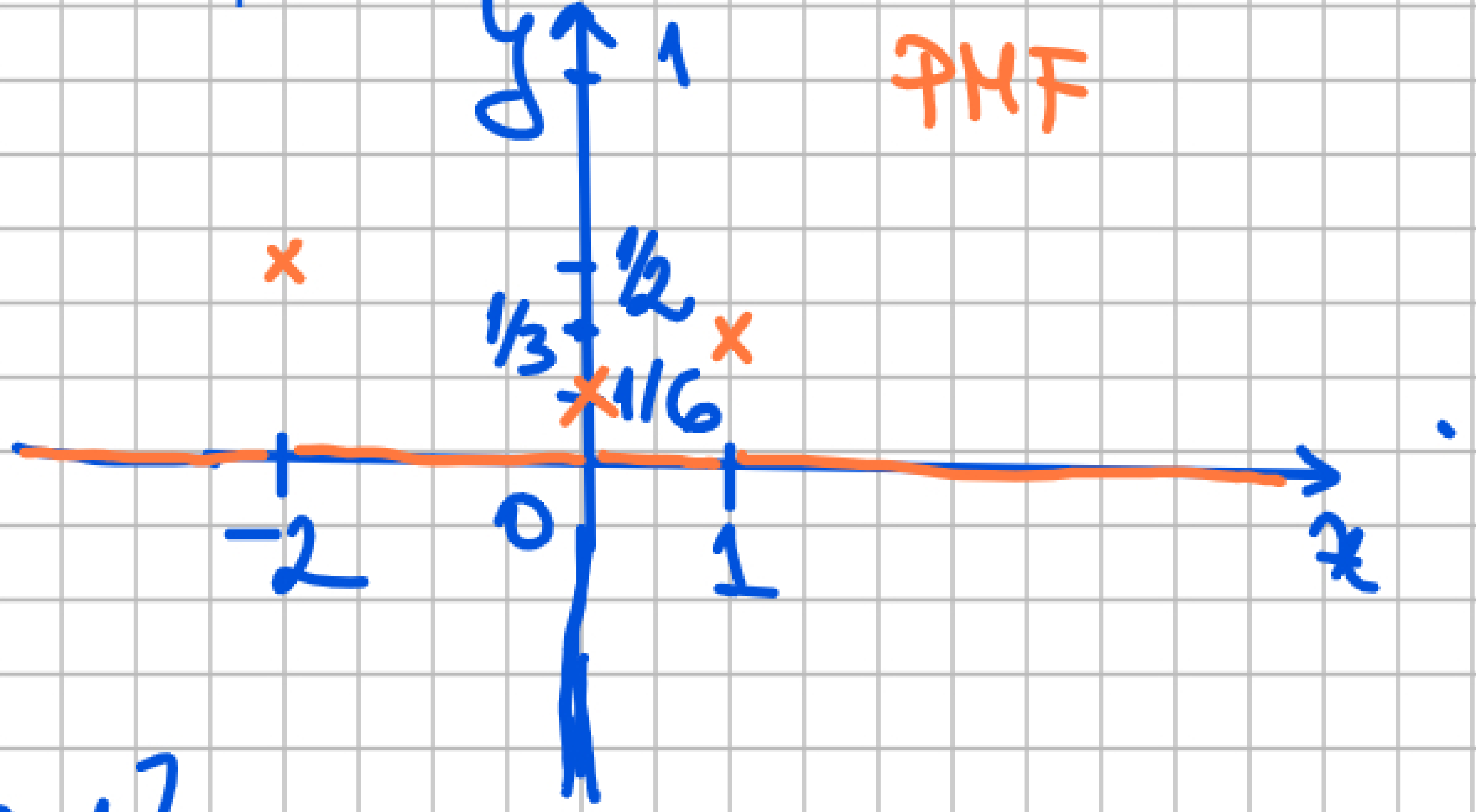
$f(x)=P(X=x)$

$f(-2)=\frac{1}{2}$

$f(0)=\frac{1}{6}$

$f(1)=\frac{1}{3}$

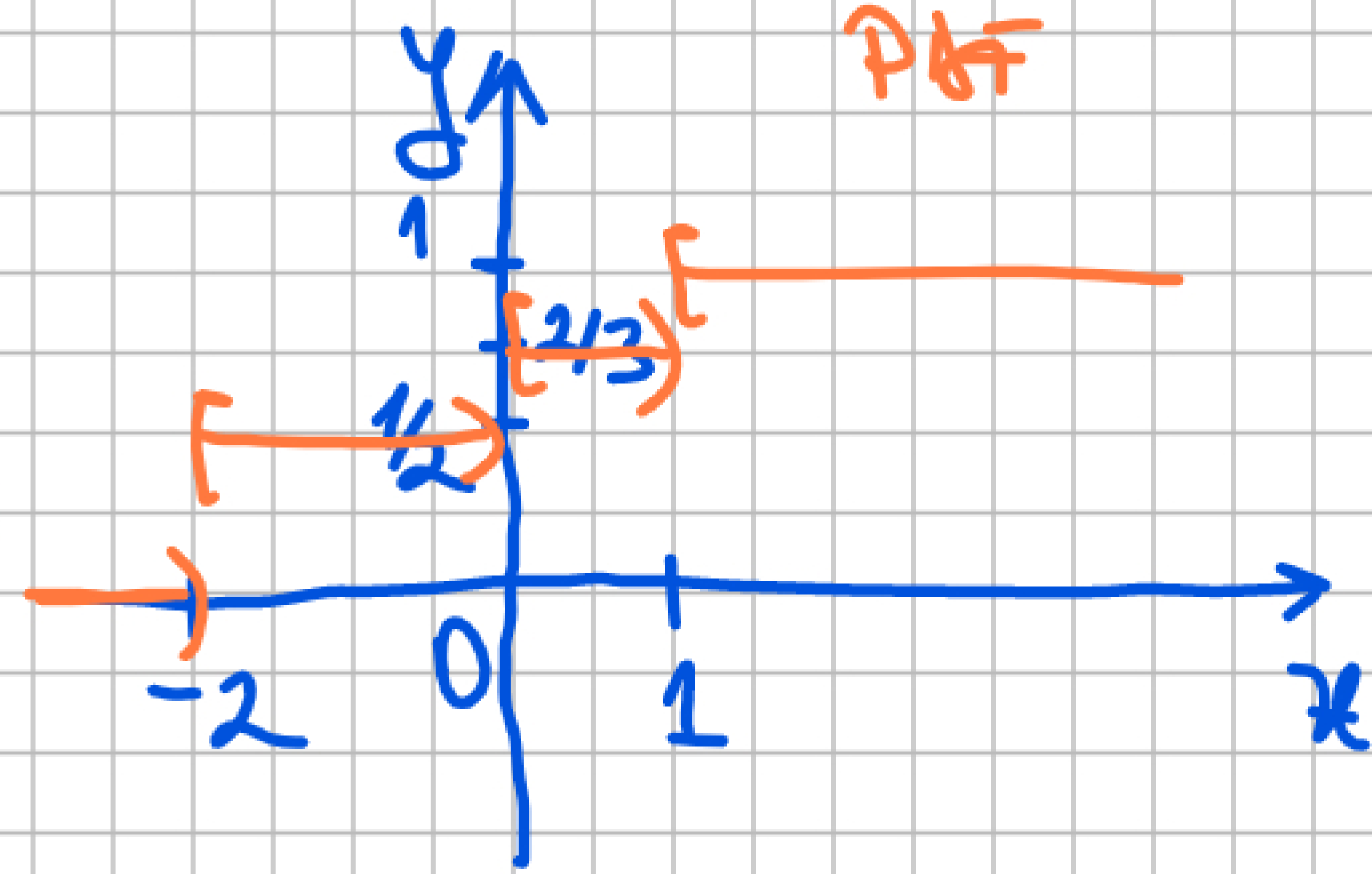
$f(x)=0,\ (\forall)\ x\notin\{-2,0,1\}$



CDF: $F:\mathbb{R}\to[0,1]$,

$F(x)=P(X\le x)$

$$F(x)=\begin{cases}0, & x<-2\\ \frac{1}{2}, & -2\le x<0\\ \frac{1}{2}+\frac{1}{6} & 0\le x<1\\ \frac{1}{2}+\frac{1}{6}+\frac{1}{3}, & x\ge 1\end{cases}=\begin{cases}0, & x<-2\\ \frac{1}{2}, & -2\le x<0\\ \frac{2}{3}, & 0\le x<1\\ 1, & x\ge 1\end{cases}$$



Compute the PMF & CDF of Y. $\longrightarrow$ homework

Example 2.3.7 (Random coin). You have one fair coin, and one biased coin which lands Heads with probability 3/4. You pick one of the coins at random and flip it three times. It lands Heads all three times. Given this information, what is the probability that the coin you picked is the fair one?

$P(A|B)$

F : the coin is fair

B : the coin is biased

3H : the coin lands Heads 3 times

$P(F)=P(B)=\frac{1}{2}$

$$P(F/3H)=\frac{P(3H/F)\cdot P(F)}{P(3H)}=\frac{P(3H/F)\cdot P(F)}{P(3H/F)\cdot P(F)+P(3H/B)\cdot P(B)}=\frac{\frac{1}{8}\cdot\cancel{\frac{1}{2}}}{\frac{1}{8}\cdot\cancel{\frac{1}{2}}+\frac{27}{64}\cdot\cancel{\frac{1}{2}}}=\frac{\frac{1}{8}}{\frac{35}{64}}=\frac{8}{35}=$$

(Bayes' rule)

$=0.229$

$P(3H)=P(3H/F)\cdot P(F)+P(3H/B)\cdot P(B)$ (law of total probability)

$P(3H|F)=\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{8}$, $P(3H/B)=\left(\frac{3}{4}\right)^3=\frac{27}{64}$

Example 2.3.9 (Testing for a rare disease). A patient named Fred is tested for a disease called conditionitis, a medical condition that afflicts 1% of the population. The test result is positive, i.e., the test claims that Fred has the disease. Let D be the event that Fred has the disease and T be the event that he tests positive.

Suppose that the test is "95% accurate"; there are different measures of the accuracy of a test, but in this problem it is assumed to mean that $P(T|D) = 0.95$ and $P(T^c|D^c) = 0.95$. The quantity $P(T|D)$ is known as the *sensitivity* or *true positive rate* of the test, and $P(T^c|D^c)$ is known as the *specificity* or *true negative rate*.

Find the conditional probability that Fred has conditionitis, given the evidence provided by the test result.

$P(\bar{A}) = 1 - P(A)$

$P(\bar{A}/B) = 1 - P(A/B)$

$P(A/B) = 1 - P(\bar{A}/B)$

$\cancel{P(A/\bar{B})} \neq 1 - P(A/B)$

D: Fred has the disease

T: Fred tests positive

$P(T/D) = 0.95$, $P(\bar{T}/\bar{D}) = 0.95$

$\downarrow$ sensitivity $\qquad \downarrow$ specificity

$P(D/T) = ?$

$$P(D/T) = \frac{P(T/D)\cdot P(D)}{P(T)} = \frac{P(T/D)\cdot P(D)}{P(T/D)\cdot P(D) + P(T/\bar{D})\cdot P(\bar{D})}$$

$P(T/\bar{D}) = 1 - P(\bar{T}/\bar{D}) = 1 - 0.95 = 0.05$

$P(D) = 0.01 \Rightarrow P(\bar{D}) = 1 - 0.01 = 0.99$

$$P(D/T) = \frac{0.95\cdot 0.01}{0.95\cdot 0.01 + 0.05\cdot 0.99} = 0.161$$

5. Three cards are dealt from a standard, well-shuffled deck. The first two cards are flipped over, revealing the Ace of Spades as the first card and the 8 of Clubs as the second card. Given this information, find the probability that the third card is an ace in two ways: using the definition of conditional probability, and by symmetry.

$\rightarrow$ homework

32. ⓢ Consider four nonstandard dice (the *Efron dice*), whose sides are labeled as follows (the 6 sides on each die are equally likely).

A: 4, 4, 4, 4, 0, 0

B: 3, 3, 3, 3, 3, 3

C: 6, 6, 2, 2, 2, 2

D: 5, 5, 5, 1, 1, 1

These four dice are each rolled once. Let A be the result for die A, B be the result for die B, etc.

(a) Find $P(A > B)$, $P(B > C)$, $P(C > D)$, and $P(D > A)$.

(b) Is the event $A > B$ independent of the event $B > C$? Is the event $B > C$ independent of the event $C > D$? Explain.

$A > B$: $A = 4, B = 3$

$P(A > B) = P(A = 4, B = 3) = P(A = 4) \cdot P(B = 3) = \frac{4}{6} \cdot 1 = \frac{2}{3}$

$B > C$: $B = 3, C = 2$

$P(B > C) = P(B = 3, C = 2) = P(B = 3) \cdot P(C = 2) = 1 \cdot \frac{4}{6} = \frac{2}{3}$

$\left.\begin{array}{l} A > B : A = 4 \\ B > C : C = 2 \end{array}\right\}$ independent

$\left.\begin{array}{l} B > C : C = 2 \\ C > D : (C = 6) \text{ or } (C = 2, D = 1) \end{array}\right\}$ not independent

- A, B - indep. $\iff P(A \cap B) = P(A) \cdot P(B)$

$P(B > C \cap C > D) = \dots$

$\neq$

$P(B > C) \cdot P(C > D) = \dots$

p. 68 → Monty Hall pb → implement the Monty Hall pb. in R
and estimate the pb. of winning in 2 cases:

1) you switch the door you initially chose
2) you stick with the initial choice.

(homework)

sgr. 1

1. Consider the random variables X and Y with the following distributions. Determine the distributions of 3X, 3X+Y, -2Y, X*Y, X^2, X/Y and compute E[X], V[X], E[2X+3Y], V[2X-3Y], P(-2<=X<0), P(Y>=1) , X and Y are independent.

$$X: \begin{pmatrix} -2 & 0 & 1 & 3 \\ \frac{1}{5} & \frac{2}{5} & \frac{1}{10} & \frac{3}{10} \end{pmatrix}, \quad Y: \begin{pmatrix} -2 & 3 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$P(X=-2, X=-2) = P(X=-2) = \frac{1}{5}$$

$g(X)$ - r.v. where $g: \mathbb{R} \to \mathbb{R}$ a function

$|X|, \ \ln(|X|), \ \exp(X)$

$$3X: \begin{pmatrix} 3\cdot(-2) & 3\cdot 0 & 3\cdot 1 & 3\cdot 3 \\ \frac{1}{5} & \frac{2}{5} & \frac{1}{10} & \frac{3}{10} \end{pmatrix}, \quad 3X: \begin{pmatrix} -6 & 0 & 3 & 9 \\ \frac{1}{5} & \frac{2}{5} & \frac{1}{10} & \frac{3}{10} \end{pmatrix}$$

$$3X+Y: \begin{pmatrix} -6+(-2) & -6+3 & 0+(-2) & 0+3 & 3+(-2) & 3+3 & 9+(-2) & 9+3 \\ \frac{1}{5}\cdot\frac{2}{3} & \frac{1}{5}\cdot\frac{1}{3} & \frac{2}{5}\cdot\frac{2}{3} & \frac{2}{5}\cdot\frac{1}{3} & \frac{1}{10}\cdot\frac{2}{3} & \frac{1}{10}\cdot\frac{1}{3} & \frac{3}{10}\cdot\frac{2}{3} & \frac{3}{10}\cdot\frac{1}{3} \end{pmatrix}$$

$$3X+Y:\begin{pmatrix} -8 & -3 & -2 & 3 & 1 & 6 & 7 & 12 \\ \frac{2}{15} & \frac{1}{15} & \frac{4}{15} & \frac{2}{15} & \frac{2}{30} & \frac{1}{30} & \frac{6}{30} & \frac{3}{30} \end{pmatrix}$$

$$6+8+4+2+\underbrace{1+9}=30$$

$$3X+Y:\begin{pmatrix} -8 & -3 & -2 & 1 & 3 & 6 & 7 & 12 \\ \frac{2}{15} & \frac{1}{15} & \frac{4}{15} & \frac{1}{15} & \frac{2}{15} & \frac{1}{30} & \frac{1}{5} & \frac{1}{10} \end{pmatrix}$$

$$\begin{matrix} 1 & 1 & 1 \\ \frac{1}{10} & \frac{1}{5} & \frac{1}{5} \end{matrix} \longrightarrow \begin{matrix} 1 \\ \frac{5}{10} \end{matrix}$$

$$4,\underbrace{-6,0},-2,3,9$$

$$-2Y:\begin{pmatrix} (-2)\cdot(-2) & (-2)\cdot 3 \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix},\ -2Y:\begin{pmatrix} -6 & 4 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

$$X*Y:\begin{pmatrix} (-2)\cdot(-2) & (-2)\cdot 3 & 0\cdot(-2) & 0\cdot(-3) & 1\cdot(-2) & 1\cdot 3 & 3\cdot(-2) & 3\cdot 3 \\ \frac{1}{5}\cdot\frac{2}{3} & \frac{1}{5}\cdot\frac{1}{3} & \frac{2}{5}\cdot\frac{2}{3} & \frac{2}{5}\cdot\frac{1}{3} & \frac{1}{10}\cdot\frac{2}{3} & \frac{1}{10}\cdot\frac{1}{3} & \frac{3}{10}\cdot\frac{2}{3} & \frac{3}{10}\cdot\frac{1}{3} \end{pmatrix}$$

$$X*Y:\begin{pmatrix} -6 & -2 & 0 & 3 & 4 & 9 \\ 4/15 & 1/15 & 2/5 & 1/30 & 2/15 & 1/10 \end{pmatrix}$$

$$\frac{4}{15}+\frac{2}{15}=\frac{6}{15}=\frac{2}{5};\quad \frac{1}{15}+{}^{3/}\frac{1}{5}=\frac{4}{15}$$

$$X|Y: \begin{pmatrix} \frac{-2}{-2} & \frac{-2}{3} & \frac{0}{-2} & \frac{0}{3} & \frac{1}{-2} & \frac{1}{3} & \frac{3}{-2} & \frac{3}{3} \\ \frac{1}{5}\cdot\frac{2}{3} & \frac{1}{5}\cdot\frac{1}{3} & \dots \end{pmatrix} \dots$$

$$P(X=-2, Y=-2) = P(X=-2 \cap Y=-2) \overset{X,Y\text{-ind.}}{=\!=} P(X=-2)\cdot P(Y=-2)$$

$$E[X] = -2\cdot\frac{1}{5} + 0\cdot\frac{2}{5} + 1\cdot\frac{1}{10} + 3\cdot\frac{3}{10} = -\frac{2}{5} + \frac{1}{10} + \frac{9}{10} = -\frac{2}{5} + \cancel{\frac{5}{1}} 1 = \frac{3}{5}$$

$$V[X] = E[X^2] - E^2[X] = \frac{18}{5} - \left(\frac{3}{5}\right)^2 = \frac{\overset{5}{18}}{5} - \frac{9}{25} = \frac{90-9}{25} = \frac{81}{25}$$

$$X^2: \begin{pmatrix} (-2)^2 & 0^2 & 1^2 & 3^2 \\ \frac{1}{5} & \frac{2}{5} & \frac{1}{10} & \frac{3}{10} \end{pmatrix}, \quad X^2: \begin{pmatrix} 0 & 1 & 4 & 9 \\ \frac{2}{5} & \frac{1}{10} & \frac{1}{5} & \frac{3}{10} \end{pmatrix}$$

$$E[X^2] = \frac{1}{10} + \frac{\overset{2}{\cancel{4}}}{5} + \frac{27}{10} = \frac{1+8+27}{10} = \frac{36}{10} = \frac{18}{5}$$

$$Std[X] = \sqrt{V[X]} = \sqrt{\frac{81}{25}} = \frac{9}{5}$$

$$E[2X+3Y] = E[2X] + E[3Y] = 2E[X] + 3E[Y] = 2\cdot\frac{3}{5} + 3\cdot\left(-\frac{1}{3}\right) = \frac{6}{5} - 1 = \frac{1}{5}$$

$$E[Y] = -2\cdot\frac{2}{3} + 3\cdot\frac{1}{3} = -\frac{1}{3}$$

$$V[2X-3Y] \overset{X,Y\text{-ind}}{=} V[2X] + V[-3Y] = 2^2 V[X] + (-3)^2 V[Y] = 4V[X] + 9V[Y] =$$

$$= 4\cdot\frac{81}{25} + 9\cdot\frac{50}{9} = \frac{244}{25} + 50 = \frac{244+1250}{25} = \frac{1494}{25}$$

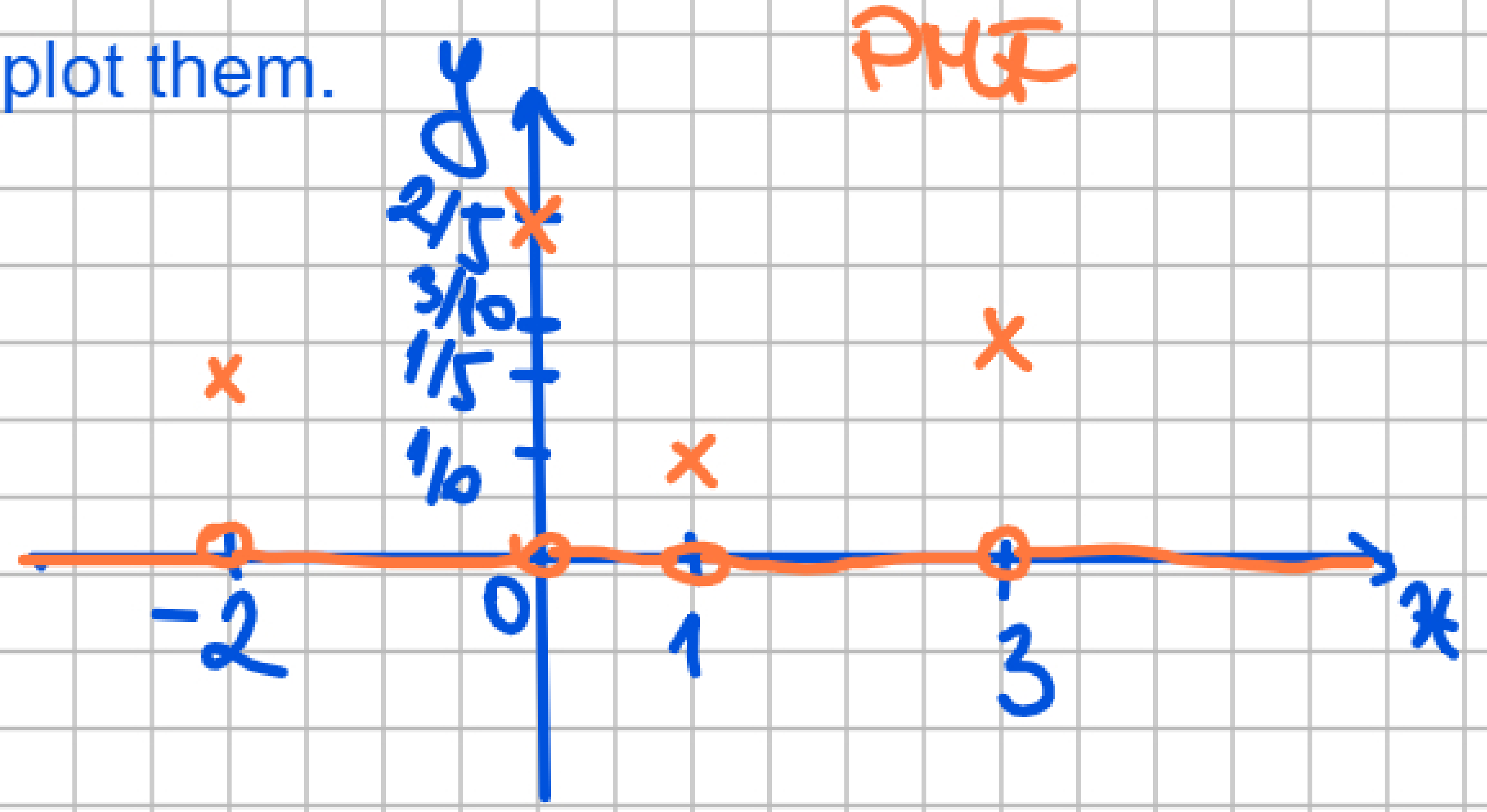
$$V[Y] = E[Y^2] - E^2[Y] = \frac{17}{3} - \left(-\frac{1}{3}\right)^2 = \frac{17}{3} - \frac{1}{9} = \frac{50}{9} > 0$$

$$E[Y^2] = (-2)^2\cdot\frac{2}{3} + 9\cdot\frac{1}{3} = \frac{8}{3} + 3 = \frac{17}{3}$$

$$P(-2 \leq X < 0) = P(X=-2) = \frac{1}{5}$$

$$P(Y \geq 1) = P(Y=3) = \frac{1}{3}$$

2. Compute the PMF and CDF of the random variables X and Y and plot them.



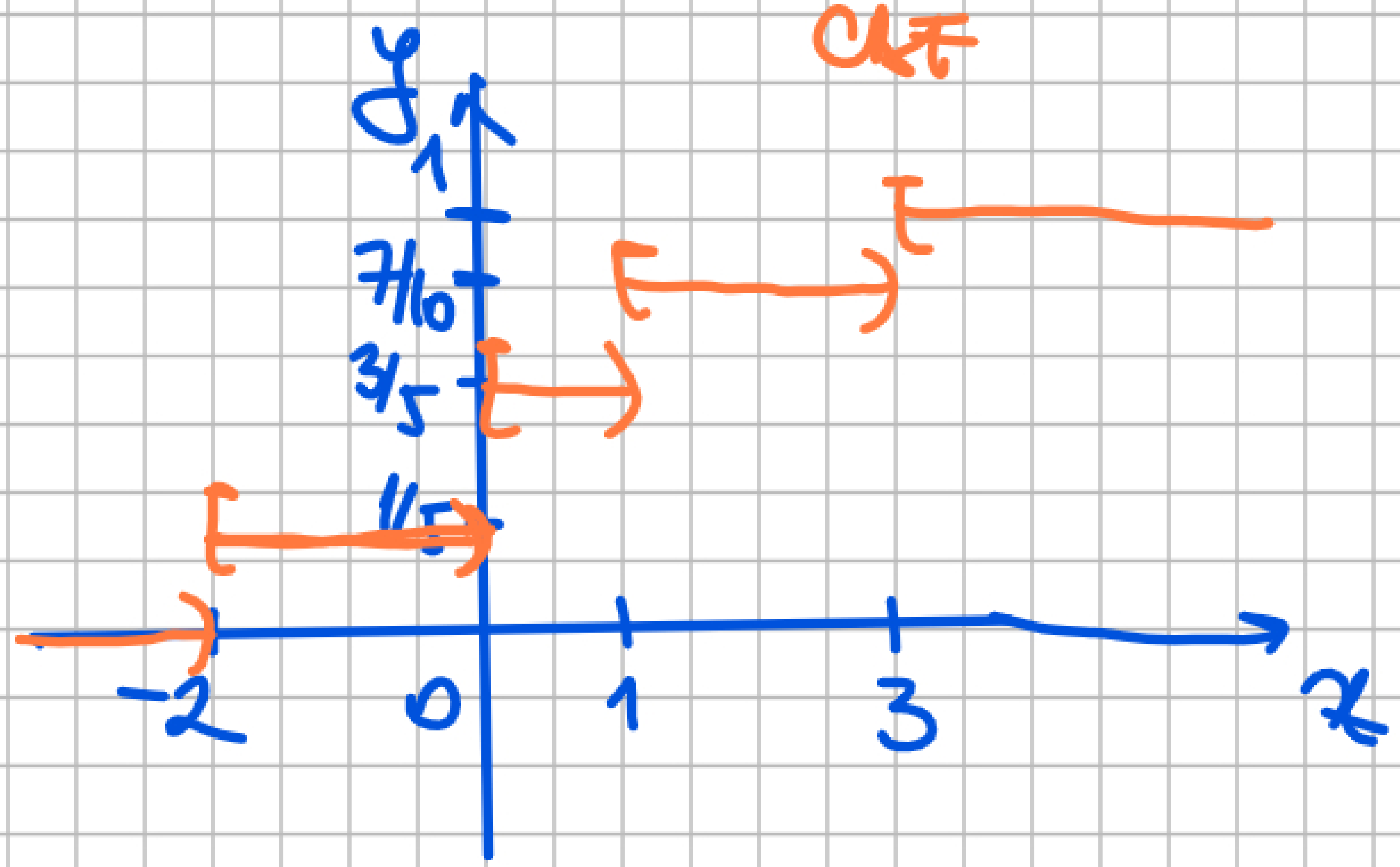
PMF : $f:\mathbb{R}\to[0,1]$, $f(x)=P(X=x)$

$f(-2)=\frac{1}{5}$, $f(0)=\frac{2}{5}$, $f(1)=\frac{1}{10}$, $f(3)=\frac{3}{10}$

$f(x)=0$, $(\forall x\notin\{-2,0,1,3\}$

CDF : $F:\mathbb{R}\to[0,1]$, $F(x)=P(X\le x)$

$$F(x)=\begin{cases}0, & x<-2\\ \frac{1}{5}, & -2\le x<0\\ \frac{1}{5}+\frac{2}{5}, & 0\le x<1\\ \frac{1}{5}+\frac{2}{5}+\frac{1}{10}, & 1\le x<3\\ \frac{1}{5}+\frac{2}{5}+\frac{1}{10}+\frac{3}{10}, & x\ge 3\end{cases} = \begin{cases}0, & x<-2\\ \frac{1}{5}, & -2\le x<0\\ \frac{3}{5}, & 0\le x<1\\ \frac{7}{10}, & 1\le x<3\\ 1, & x\ge 3\end{cases}$$



Homework : PMF & CDF of Y

12. Alice is trying to communicate with Bob, by sending a message (encoded in binary) across a channel.

(a) Suppose for this part that she sends only one bit (a 0 or 1), with equal probabilities. If she sends a 0, there is a 5% chance of an error occurring, resulting in Bob receiving a 1; if she sends a 1, there is a 10% chance of an error occurring, resulting in Bob receiving a 0. Given that Bob receives a 1, what is the probability that Alice actually sent a 1?

(b) To reduce the chance of miscommunication, Alice and Bob decide to use a *repetition code*. Again Alice wants to convey a 0 or a 1, but this time she repeats it two more times, so that she sends 000 to convey 0 and 111 to convey 1. Bob will decode the message by going with what the majority of the bits were. Assume that the error probabilities are as in (a), with error events for different bits independent of each other. Given that Bob receives 110, what is the probability that Alice intended to convey a 1?

A_0 : Alice sends a 0

A_1 : Alice sends a 1

B_0 : Bob receives a 0

B_1 : Bob receives a 1

a) $P(B_1|A_0) = 0.05$ $\quad \underbrace{P(A_0)=P(A_1)}_{\text{prior probabilities}}=0.5$

$P(B_0|A_1) = 0.1$

$P(A_1|B_1) = ?$ → evidence

$$\underbrace{P(A_1|B_1)}_{\text{posterior probab.}} \overset{\text{Bayes' rule}}{=} \frac{P(B_1|A_1)\cdot P(A_1)}{P(B_1)} \overset{\text{law of total probability}}{=} \frac{P(B_1|A_1)\cdot P(A_1)}{P(B_1|A_1)P(A_1)+P(B_1|A_0)P(A_0)} = \frac{0.9\cdot \cancel{0.5}}{0.9\cdot\cancel{0.5}+0.05\cdot\cancel{0.5}} =$$

$$= \frac{0.9}{0.95} = \frac{90}{95} = 0.947$$

$P(B_1|A_1) = 1-P(B_0|A_1) = 1-0.1=0.9$

$B_1 = \overline{B_0}$, $P(\overline{A})=1-P(A)$; $P(\overline{A}|B)=1-P(A|B)$

$A_1, A_2, \dots, A_n$ – partition of the sample space Ω : $\begin{cases} A_i \cap A_j = \emptyset \quad (\forall)\, i \neq j \\ A_1 \cup A_2 \cup \dots \cup A_n = \Omega \end{cases}$

$$P(X) = P(X|A_1) \cdot P(A_1) + P(X|A_2) \cdot P(A_2) + \dots + P(X|A_n) P(A_n)$$

b) $\longrightarrow$ homework

10. Fred is working on a major project. In planning the project, two milestones are set up, with dates by which they should be accomplished. This serves as a way to track Fred's progress. Let A_1 be the event that Fred completes the first milestone on time, A_2 be the event that he completes the second milestone on time, and A_3 be the event that he completes the project on time.

Suppose that $P(A_{j+1}|A_j) = 0.8$ but $P(A_{j+1}|A_j^c) = 0.3$ for $j = 1, 2$, since if Fred falls behind on his schedule it will be hard for him to get caught up. Also, assume that the second milestone supersedes the first, in the sense that once we know whether he is on time in completing the second milestone, it no longer matters what happened with the first milestone. We can express this by saying that A_1 and A_3 are conditionally independent given A_2 and they're also conditionally independent given A_2^c.

(a) Find the probability that Fred will finish the project on time, given that he completes the first milestone on time. Also find the probability that Fred will finish the project on time, given that he is late for the first milestone.

(b) Suppose that $P(A_1) = 0.75$. Find the probability that Fred will finish the project on time.

20. The Jack of Spades (with cider), Jack of Hearts (with tarts), Queen of Spades (with a wink), and Queen of Hearts (without tarts) are taken from a deck of cards. These four cards are shuffled, and then two are dealt. Note: Literary references to cider, tarts, and winks do not need to be considered when solving this problem.

(a) Find the probability that both of these two cards are queens, given that the first card dealt is a queen.

(b) Find the probability that both are queens, given that at least one is a queen.

(c) Find the probability that both are queens, given that one is the Queen of Hearts.

$\rightarrow$ homework

1. People are arriving at a party one at a time. While waiting for more people to arrive they entertain themselves by comparing their birthdays. Let X be the number of people needed to obtain a birthday match, i.e., before person X arrives no two people have the same birthday, but when person X arrives there is a match. Find the PMF of X.

H: Implement the Monty Hall pb. in R and estimate the probability of winning in 2 cases:

1) you switch to the other unopened door
2) you stick with the initial option

sgr. 4

1. Let X and Y be two independent random variables with the following distributions. Determine the distributions if X+Y, 1/3*X, X*Y, X\Y, X^2, |X| and compute E[X], V[X], E[2X-Y], V[-X+3Y], P(-1<=X<=1), P(Y>=0).

$$X:\begin{pmatrix} -1 & 0 & 1 & 2 \\ \frac{1}{12} & \frac{1}{4} & \frac{1}{6} & \frac{1}{2} \end{pmatrix} \qquad Y:\begin{pmatrix} 1 & 2 \\ \frac{2}{5} & \frac{3}{5} \end{pmatrix}$$

$$X+Y:\begin{pmatrix} -1+1 & -1+2 & 0+1 & 0+2 & 1+1 & 1+2 & 2+1 & 2+2 \\ \frac{1}{30} & \frac{1}{20} & \frac{1}{\cancel{4}_{2}}\cdot\frac{\cancel{2}}{5} & \frac{1}{4}\cdot\frac{3}{5} & \frac{1}{\cancel{6}}\cdot\frac{\cancel{2}}{5} & \frac{1}{\cancel{6}}\cdot\frac{\cancel{3}}{5} & \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{5} & \frac{1}{2}\cdot\frac{3}{5} \end{pmatrix}$$

$$P(X=-1, Y=1) \overset{X,Y-ind}{=\!=} P(X=-1)\cdot P(Y=1) = \frac{1}{\cancel{12}}\cdot\frac{\cancel{2}}{5} = \frac{1}{30}$$

$$P(X=-1, Y=2) = P(X=-1)\cdot P(Y=2) = \frac{1}{12}\cdot\frac{3}{5} = \frac{1}{20}$$

$$X+Y:\begin{pmatrix} 0 & -1 & 1 & 2 & 2 & 3 & 3 & 4 \\ \frac{1}{30} & \frac{1}{20} & \frac{1}{10} & \frac{3}{20} & \frac{1}{15} & \frac{1}{10} & \frac{1}{5} & \frac{3}{10} \end{pmatrix} \quad \boxed{X+Y}\begin{pmatrix} -1 & 0 & 1 & 2 & 3 & 4 \\ \frac{1}{20} & \frac{1}{30} & \frac{1}{10} & \frac{13}{60} & \frac{3}{10} & \frac{3}{10} \end{pmatrix}$$

$$\frac{3}{20}^{\,3/} + \frac{1}{15}^{\,4/} = \frac{13}{60}\ ; \quad \frac{1}{10} + \frac{1}{5}^{\,2/} = \frac{3}{10}$$

$$3+2+6+13+18+18 = 24+36 = 60 \checkmark$$

$\frac{1}{3}\cdot X: \begin{pmatrix} \frac{1}{3}\cdot(-1) & \frac{1}{3}\cdot 0 & \frac{1}{3}\cdot 1 & \frac{1}{3}\cdot 2 \\ \frac{1}{12} & \frac{1}{4} & \frac{1}{6} & \frac{1}{2} \end{pmatrix}, \quad \frac{1}{3}X: \begin{pmatrix} -\frac{1}{3} & 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{12} & \frac{1}{4} & \frac{1}{6} & \frac{1}{2} \end{pmatrix}$

$$X*Y: \begin{array}{c} \begin{array}{cccccccc} -1 & -2 & 0 & 0 & 1 & 2 & 2 & 4 \end{array} \\ \begin{pmatrix} (-1)\cdot 1 & (-1)\cdot 2 & 0\cdot 1 & 0\cdot 2 & 1\cdot 1 & 1\cdot 2 & 2\cdot 1 & 2\cdot 2 \\ \frac{1}{12}\cdot\frac{2}{5} & \frac{1}{12}\cdot\frac{3}{5} & \frac{1}{4}\cdot\frac{2}{5} & \frac{1}{4}\cdot\frac{3}{5} & \frac{1}{6}\cdot\frac{2}{5} & \frac{1}{6}\cdot\frac{3}{5} & \frac{1}{2}\cdot\frac{2}{5} & \frac{1}{2}\cdot\frac{3}{5} \end{pmatrix} \end{array}$$

$$P(X=-1, Y=1) = P(X=-1)\cdot P(Y=1)$$
$$= \frac{1}{12}\cdot\frac{2}{5} = \frac{1}{6}$$

$\boxed{X*Y}: \begin{pmatrix} -2 & -1 & 0 & 1 & 2 & 4 \\ \frac{1}{20} & \frac{1}{30} & \frac{1}{4} & \frac{1}{15} & \frac{3}{10} & \frac{3}{10} \end{pmatrix}, \quad \boxed{X|Y}: \begin{pmatrix} \frac{-1}{1} & \frac{-1}{2} & \frac{0}{1} & \frac{0}{2} & \dots \\ \frac{1}{12}\cdot\frac{2}{5} & \frac{1}{12}\cdot\frac{3}{5} & \frac{1}{4}\cdot\frac{2}{5} & \frac{1}{4}\cdot\frac{3}{5} & \dots \end{pmatrix}$

$X^2: \begin{pmatrix} (-1)^2 & 0^2 & 1^2 & 2^2 \\ \frac{1}{12} & \frac{1}{4} & \frac{1}{6} & \frac{1}{2} \end{pmatrix}, \quad \boxed{X^2}: \begin{pmatrix} 0 & 1 & 4 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$

$|X|: \begin{pmatrix} |-1| & |0| & |1| & |2| \\ \frac{1}{12} & \frac{1}{4} & \frac{1}{6} & \frac{1}{2} \end{pmatrix}, \quad \boxed{|X|}: \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$

$$X: \begin{pmatrix} -1 & 0 & 1 & 2 \\ \frac{1}{12} & \frac{1}{4} & \frac{1}{6} & \frac{1}{2} \end{pmatrix} \qquad E[X] = (-1)\cdot\frac{1}{12} + 0\cdot\frac{1}{4} + 1\cdot\frac{1}{6} + 2\cdot\frac{1}{2} = -\frac{1}{12} + \frac{1}{6} + 1 = \frac{13}{12} = 1.083$$

$$V[X] = E[X^2] - E^2[X] = \frac{9}{4} - \left(\frac{13}{12}\right)^2 = \frac{9}{4} - \frac{169}{144} = \frac{324-169}{144} = \frac{155}{144} = 1.076$$

$$E[2X-Y] = E[2X] + E[-Y] = 2E[X] - E[Y] = 2\cdot\frac{13}{12} - \frac{8}{5} = \frac{13}{6} - \frac{8}{5} = \frac{65-48}{30} = \frac{17}{30}$$

$$V[-X+3Y] \overset{X,Y\text{-ind}}{=} V[-X] + V[3Y] = (-1)^2 V[X] + 3^2 V[Y] = V[X] + 9V[Y] =$$

$$= \frac{155}{144} + 9\cdot\frac{6}{25} = \frac{11651}{25\cdot 144} = \frac{11651}{3600} = 3.236$$

$$V[Y] = E[Y^2] - E^2[Y] = \boxed{\frac{14}{5}} - \left(\frac{8}{5}\right)^2 = \frac{14}{5} - \frac{64}{25} = \frac{70-64}{25} = \frac{6}{25}$$

$$P(-1 \le X \le 1) = P(X=-1) + P(X=0) + P(X=1) = \frac{1}{12} + \frac{1}{4} + \frac{1}{6} = 1 - \frac{1}{2} = \frac{1}{2}$$

$$P(Y > 0) = P(Y=1) + P(Y=2) = 1$$

$$P\left(X \le -\frac{1}{2}\right) = P(X=-1) = \frac{1}{12}$$

$$P\left(X \le \frac{1}{2}\right) = \frac{1}{12} + \frac{1}{4} = \frac{4}{12} = \frac{1}{3}$$

$$P(X \le x) = \frac{1}{12} + \frac{1}{4} + \frac{1}{6} \quad x \in [1,2)$$

$$P(X \le 3) = 1 \quad x \ge 2$$

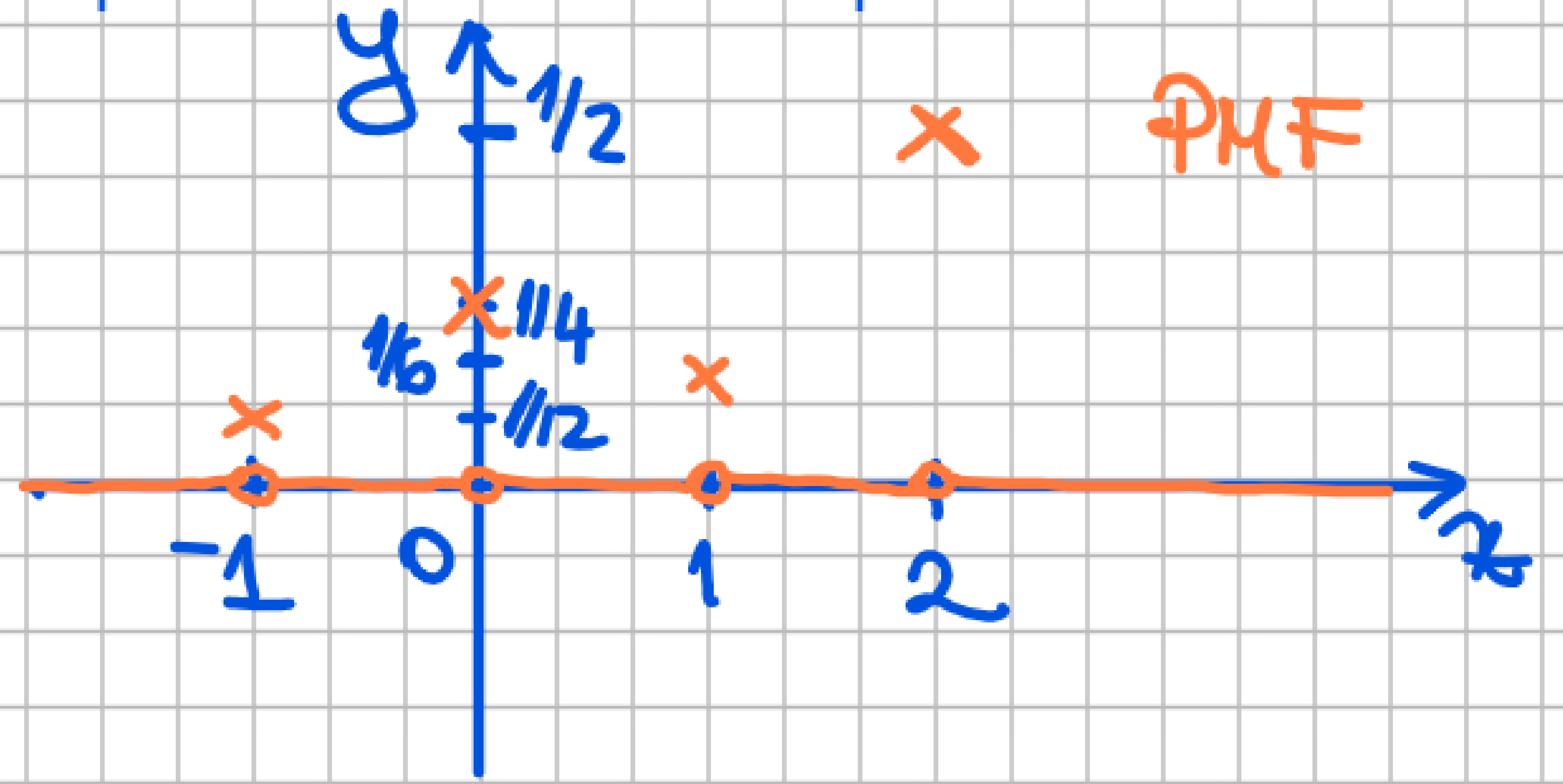
2. Compute the PMF and CDF of the random variables X and Y from the previous exercise and plot them.

PMF: $f:\mathbb{R}\to[0,1]$, $f(-1)=\frac{1}{12}$, $f(0)=\frac{1}{4}$

$f(x)=P(X=x)$

$f(1)=\frac{1}{6}$, $f(2)=\frac{1}{2}$

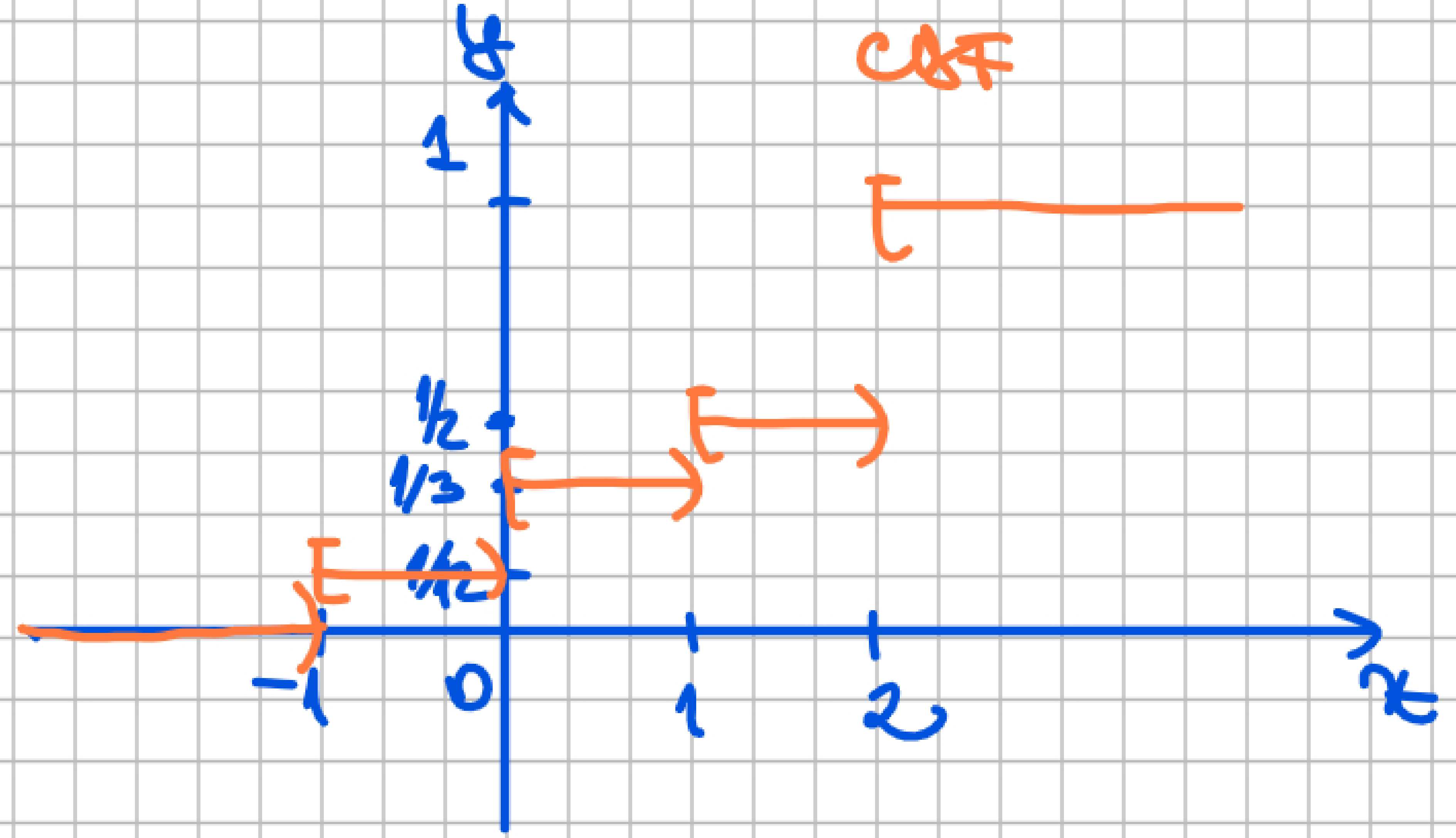
$f(x)=0,\ (\forall)\, x\notin\{-1,0,1,2\}$



CDF: $F:\mathbb{R}\to[0,1]$

$F(x)=P(X\le x)$

$$F(x)=\begin{cases}0, & x<-1\\ \frac{1}{12}, & -1\le x<0\\ \frac{1}{12}+\frac{1}{4}, & 0\le x<1\\ \frac{1}{12}+\frac{1}{4}+\frac{1}{6}, & 1\le x<2\\ \frac{1}{12}+\frac{1}{4}+\frac{1}{6}+\frac{1}{2}, & x\ge 2\end{cases}=\begin{cases}0, & x<-1\\ \frac{1}{12}, & -1\le x<0\\ \frac{1}{3}, & 0\le x<1\\ \frac{1}{2}, & 1\le x<2\\ 1, & x\ge 2\end{cases}$$



Homework : PMF + CDF of Y & plot them

21. A fair coin is flipped 3 times. The toss results are recorded on separate slips of paper (writing "H" if Heads and "T" if Tails), and the 3 slips of paper are thrown into a hat.

(a) Find the probability that all 3 tosses landed Heads, given that at least 2 were Heads.

(b) Two of the slips of paper are randomly drawn from the hat, and both show the letter H. Given this information, what is the probability that all 3 tosses landed Heads?

$P(A|B) = \frac{P(A\cap B)}{P(B)}$

"A given B"

B → has occured

$A_1, A_2, \dots, A_m$ – partition of Ω

$P(B) = P(B|A_1)\cdot P(A_1) + P(B|A_2)P(A_2) + \dots + P(B|A_n)P(A_m)$ – law of total probability

$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B)}$, $i=\overline{1,m}$ Bayes' rule

HHT
HTH
THH

(a) $P(A/B) = \frac{P(A\cap B)}{P(B)} = \frac{\frac{1}{8}}{1/2} = \frac{1}{8}\cdot 2 = \frac{1}{4}$

A : all 3 tosses landed Heads, $A = \{HHH\}$

B : at least 2 tosses were Heads (at least 2 Heads), $B = \{HHT, HTH, THH, HHH\}$

$P(A) = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2^3} = \frac{1}{8}$, $P(B) = P(\text{"exactly 2H"}) + \underbrace{P(\text{"3 Heads"})}_{=\frac{1}{8}} = \frac{3}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$

$A\cap B = \{HHH\} = A$

b) $P(B|A) = \dfrac{P(A|B)\cdot P(B)}{P(A)} = \dfrac{\frac{1}{4}\cdot\frac{1}{2}}{\frac{1}{8}} = 1$

25. Ⓢ A crime is committed by one of two suspects, A and B. Initially, there is equal evidence against both of them. In further investigation at the crime scene, it is found that the guilty party had a blood type found in 10% of the population. Suspect A does match this blood type, whereas the blood type of Suspect B is unknown.

(a) Given this new information, what is the probability that A is the guilty party?

(b) Given this new information, what is the probability that B's blood type matches that found at the crime scene?

$P(M) = 0.1$

→ homework

A : A is the guilty party

B : B is the guilty party $(\overline{B} = A)$

$P(A) = P(B) = \frac{1}{2}$

M : A matches the blood type found at the crime scene

$$P(A/M) = \frac{P(M|A) \cdot P(A)}{P(M)}$$

$$P(M) = P(M|A) \cdot P(A) + P(M|B) \cdot P(B)$$

H: Simulate the Monty Hall (p.68) in R?

26. Ⓢ To battle against spam, Bob installs two anti-spam programs. An email arrives, which is either legitimate (event L) or spam (event L^c), and which program j marks as legitimate (event M_j) or marks as spam (event M_j^c) for $j \in \{1, 2\}$. Assume that 10% of Bob's email is legitimate and that the two programs are each "90% accurate" in the sense that $P(M_j|L) = P(M_j^c|L^c) = 9/10$. Also assume that given whether an email is spam, the two programs' outputs are conditionally independent.

(a) Find the probability that the email is legitimate, given that the 1st program marks it as legitimate (simplify).

(b) Find the probability that the email is legitimate, given that both programs mark it as legitimate (simplify).

7. A certain small town, whose population consists of 100 families, has 30 families with 1 child, 50 families with 2 children, and 20 families with 3 children. The *birth rank* of one of these children is 1 if the child is the firstborn, 2 if the child is the secondborn, and 3 if the child is the thirdborn.

(a) A random family is chosen (with equal probabilities), and then a random child within that family is chosen (with equal probabilities). Find the PMF, mean, and variance of the child's birth rank.

(b) A random child is chosen in the town (with equal probabilities). Find the PMF, mean, and variance of the child's birth rank.